Development and analysis of mathematical models for assessing impact of three key changes in Japanese city gas industry

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Development and Analysis of Mathematical Models for Assessing Impact of Three Key Changes in Japanese City Gas Industry

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Abstract

During the past half century, the Japanese city gas industry has experienced three key changes : 1) shift from manufactured-gas to natural gas; 2) deregulation; and 3) emergence of the all-electric house market. These changes impacted the way the city gas business had been managed in Japan. The purpose of this thesis is to develop and analyze mathematical models for capturing the impact of the three changes.

In Chapter 2, the driving forces behind the three changes are analyzed and new strategic directions resulting from these changes are discussed. This prepares a basis for developing mathematical models in subsequent chapters. Concerning the shift from manufactured-gas to natural gas, Chapter 3 establishes a methodological approach for assessing the performance of a co-generation system in terms of energy-saving. One of the advantages of natural gas over manufactured-gas can be found in the enhanced efficiency of energy transportation. In order to benefit from this advantage, it is crucial to facilitate the shift from manufactured-gas to natural gas swiftly. For this purpose, Japanese city gas companies have been keen to develop the market for co-generation systems so as to expand the demand for natural gas. The methodological approach proposed in Chapter 3 provides a useful strategic tool for disseminating co-generation systems.

Chapter 4 deals with new competitive features of the Japanese gas industry arising from deregulation. The Nash equilibrium structure is expressed explicitly for the case of M suppliers and N large-scale customers. It is shown that this game has the unique Nash equilibrium of pure strategy type under the condition that each supplier secures a set of near customers in an exclusive manner. If the delivery service areas of the two suppliers overlap each other, there exists no equilibrium within pure strategies. In Chapter 5, the competitive market model of Chapter 4 is again addressed, where the optimal pricing strategy

among mixed strategies is analyzed. In order to assure analytical tractability, we limit ourselves to two suppliers and two customers with complete symmetry. The two types of Nash equilibriums are constructed explicitly when mixed strategies are defined on a finite set of L discrete points that are chosen in such a way that their reciprocals are equally distanced in a finite interval. The limiting strategies as $L \to \infty$ are also derived explicitly. It is shown that these limiting strategies are Nash equilibriums within the context of mixed strategies defined on continuum.

In order to respond to the growing emergence of all-electric house systems, it is quite important to establish an economically viable investment strategy for installing gas pipelines in a new residential area. This problem is addressed in Chapter 6, where a newly developed residential area is formulated as a finite Markov chain in continuous time, describing moving-in and moving-out behaviors of customers. The revenue of a gas company is expressed as a reward process defined on the Markov chain, which would be affected by the number of gas pipelines installed in the beginning as well as how customers may or may not choose an all-electric house system. A computational algorithm is developed for evaluating the expected revenue at time T, enabling one to find the optimal strategy for installing gas pipelines.

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Chapter 1

Introduction

Japanese city gas industry has been growing steadily during the past half century. As a major part of the driving force for sustaining this growth, three key changes took place, which restructured the industry through several stages. The three changes are: 1) shift from manufactured-gas to natural gas; 2) deregulation; and 3) emergence of the all-electric house market. These changes impacted the way the city gas business had been managed in Japan significantly.

The city gas used to be manufactured from petroleum fuel or coal through the process of heating or distillation, and the resulting product was called the manufactured-gas. On the other hand, the natural gas was imported in the form of LNG (Liquefied Natural Gas), which was manufactured from the original gas by removing the sulfur component through the process of liquefaction before the import. Concerning the first change of the business environment, Japanese gas companies began to introduce natural gas in the middle of 1960's. The shift from manufactured-gas to natural gas was accelerated throughout 1970's, and manufactured-gas has been almost eliminated from the market by now.

The reason behind the shift from manufactured-gas to natural gas can be found in that the natural gas has three significant advantages over the manufactured-gas: 1) enhanced safety; 2) less environmentally hazardous exhausts; and 3) energy efficiency. In particular, the third factor provides the direct business advantage. More specifically, the calorific value of natural gas is twice as large as that of manufactured-gas. Accordingly, natural gas can provide city gas companies with much enhanced transportation efficiency. This advantage urged the gas companies to develop new markets, such as industrial furnace, air-conditioning of commercial buildings and co-generation. The contribution from the new markets to the profits and sales volume of the gas companies has been increasing continuously until now.

Deregulation of the gas industry has been taking place in most of developed countries in the world by now, led by the United States with the Natural Gas Policy Act (NGPA) enacted in 1978. In parallel with this global trend, the gas utilities in Japan were liberated in March 1990, resulting in the second change of the business environment. Through this deregulation in Japan, large-scale customers with consumption of more than 2 million (m^3/year) of city gas were allowed to choose a supplier freely and price regulations were abolished for these customers. So as to motivate new entrants, the existing city gas companies were forced to provide them with transportation services at regulated prices, although new entrants were still responsible for procuring natural gas by themselves. The requirement to be a large-scale customer has been relaxed step-by-step, and the requirement was set to have annual consumption of 100 thousand (m^3/year) or more in April 2007. While the price deregulation to the large-scale customers has motivated new entrants, represented by Tokyo Electric Power Company and Kansai Electric Power Company, to compete in the city gas market, it has also facilitated the existing city gas companies to begin new businesses including sales to large-scale customers through LNG lorry transportation. Accordingly, the pricing strategy has become one of the most important marketing considerations for the city gas companies.

While the deregulation has changed the industrial market as well as the large-scale commercial market substantially, the residential market was not affected much by the deregulation and remained intact. However, upon entering the new century, the emergence of all-electric house market has been changing these two markets substantially, causing the third change of the business environment. In an all-electric house system, energy demand for cooking and hot water within household is supplied totally by electricity with IH cooking heaters and an electric "heat-pump water heating and supply system." While these technologies are not new, all-electric house systems become popular because of enhancement of price-performance of those equipment, combined with the strong promotion campaigns by electricity companies. Along with the dissemination of all-electric houses, the city gas companies now face severe competitions against electricity companies in domestic energy services.

The purpose of this thesis is to develop and analyze mathematical models for capturing the impact of the three changes in the Japanese gas industry discussed above. It is intended to establish a foundation for helping Japanese city gas companies to develop new strategies for dealing with such changes. The thesis is structured as follows. Chapter 2 discusses how the three changes restructured the Japanese gas industry in detail. In particular, the driving forces behind the three changes are explained and new strategic directions resulting from the three changes are discussed. This chapter is supposed to prepare a basis for developing mathematical models in subsequent chapters. In Chapter 3, a methodological approach is established for assessing the performance of a co-generation system using natural gas in terms of energy-saving. The model can be used as a useful strategic tool for disseminating co-generation systems.

Chapter 4 deals with new competitive features of the Japanese gas industry arising from deregulation. Focusing on the price deregulation, the Nash equilibrium structure is expressed explicitly for the case of M suppliers and N large-scale customers. When the strategy space for possible price choices is finite and discrete, and only the pure strategies are considered, it is also shown, with $M = 2$, that no Nash equilibrium exists and the two suppliers continue to decrease their price in turn to the lower limit, followed by the sudden jump to the upper limit in a cyclic manner. In order to avoid this cyclic phenomenon, the case of the mixed strategies over the continuum strategy space is analyzed in Chapter 5. Approximating the continuum strategy space by a set of L points in such a way that their reciprocals are separated with equal distance, and considering the mixed strategies over the approximated discrete strategy space, the two types of Nash equilibriums are constructed explicitly. Furthermore, the two Nash equilibriums for the continuum strategy space are also derived explicitly by letting $L \to \infty$.

Addressed in Chapter 6 is the problem of how to determine the optimal number of gas pipelines in a newly developed residential area in face of the challenge from electric power companies through all-electric house system. Prior to emergence of all-electric house system, the city gas companies could count on the residential gas demand of a house located near a gas pipeline network for a very long time. Recently, however, some customers may choose all-electric house system even when the access to a gas pipeline network is available. In order to capture this competitive phenomenon, a newly developed residential area is formulated by a finite Markov chain in continuous time, where the revenue of a city gas company is described as a reward process defined on the Markov chain. Based on the uniformization procedure of Keilson [10], a computational algorithm is developed for evaluating the expected revenue of the city gas company over a finite planning horizon T as a function of the number of gas pipelines installed in the beginning. This in turn enables one to find the optimal strategy for installing gas pipelines through numerical exploration.

Chapter 2

Three Key Changes that Restructured Japanese City Gas Industry

2.1 Shift from Manufactured-Gas to Natural Gas and Related Strategic Actions

Japan's city gas industry has been growing steadily during the past half century. Figure 2.1.1 exhibits the city gas sales volume by sector in Japan, every 5 years from 1965 to 2005. It

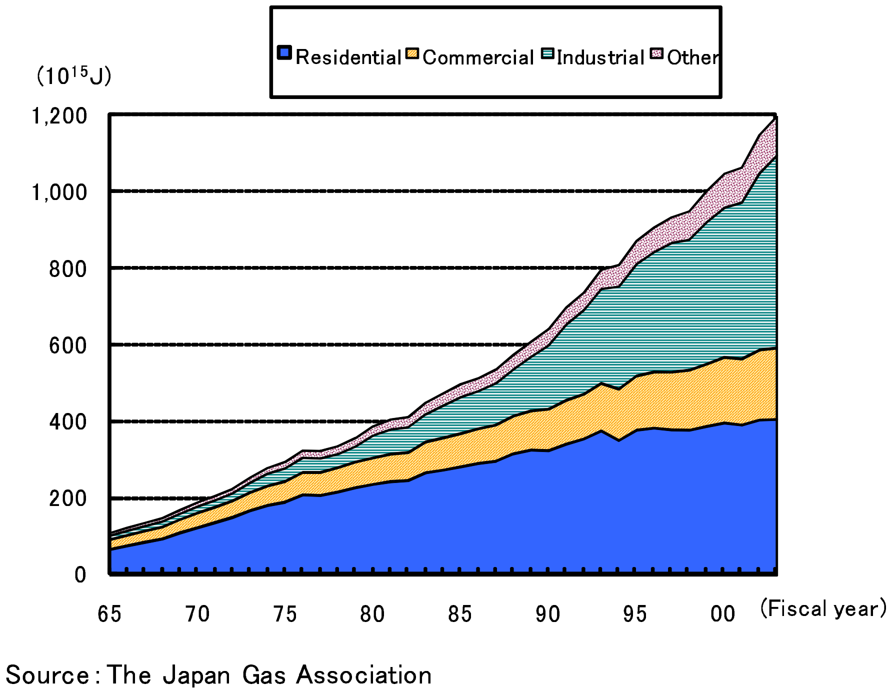

Figure 2.1.1: City Gas Sales Volume by Sector in Japan

should be noted that the core of the city gas business has been shifting from the residential sector to the industrial and commercial sectors, exceeding 60% of the total city gas sales volume in 2010. The growth rate of each of the two sectors has been much larger than that of any other sector, as can be seen in Table 2.1.1. This shift was triggered by introduction

Gas Sales Volume by Sector (10^9kcal)					Annual rate of increase			
Fiscal Year	Residential	Commercial	Industrial	Other	Residential	Commercial	Industrial	Other
1965	12,228	4,390	2,606	1,293	–	–	–	–
1970	29,439	8,777	4,846	2,651	19.21%	14.86%	13.21%	15.45%
1975	45,418	12,561	8,942	4,045	9.06%	7.43%	13.04%	8.82%
1980	56,464	16,186	14,697	5,674	4.45%	5.20%	10.45%	7.00%
1985	67,419	20,524	23,268	7,995	3.61%	4.86%	9.62%	7.10%
1990	77,602	25,662	40,261	10,142	2.85%	4.57%	11.59%	4.87%
1995	90,423	33,544	70,005	14,584	3.11%	5.50%	11.70%	7.54%
2000	94,816	40,528	93,565	21,265	0.95%	3.86%	5.97%	7.83%
2005	99,325	49,013	147,689	28,564	0.93%	3.88%	9.56%	6.08%

(Source: The Japan Gas Association, Gas Industry BINRAN (in Japanese))

Table 2.1.1: Rate of Increase in Gas Sales Volume by Sector in Japan

of natural gas into the city gas industry in early 1960's.

Prior to the natural gas, the city gas was manufactured from petroleum fuel or coal through the process of heating or distillation, and the resulting product was called the manufactured-gas. On the other hand, the natural gas was imported in the form of LNG (Liquefied Natural Gas), which was manufactured from the original gas by removing the sulfur component through the process of liquefaction before the import. As we will see, the natural gas has several significant advantages over the manufactured-gas and the former has almost eliminated the latter from the market by now.

For city gas companies, the shift from the manufactured-gas to the natural gas was quite challenging. Apparently, Japan has no natural gas resources, and the natural gas had to be imported in the form of LNG. Since the huge amount of investment would be needed for developing the liquefaction plant to produce LNG, it was absolutely necessary for potential exporting countries to secure long-term buyers. However, LNG was not traded on commercial basis at that time, and Japanese city gas companies, in collaboration with Japanese electric power companies, had to create the LNG market from the very beginning. The market was established in 1969. Japan is now the largest importer of LNG in the world, having 28 LNG terminals within Japan and trades about 100 million ton LNG per year.

In order to introduce the natural gas into the market, another obstacle had to be overcome. The burning rate of the natural gas is much higher than that of the manufactured-gas. Accordingly, so as to replace the manufactured-gas by the natural gas, all stove burners and any other gas equipment had to be adjusted, e.g. replacing burning parts. Workers had to visit every customer one by one for making necessary adjustments. It was quite time-consuming, laborious and costly.

Despite these difficulties, the shift from the manufactured-gas to the natural gas took place because the natural gas has three significant advantages over the manufactured-gas: 1) enhanced safety; 2) less environmentally hazardous exhausts; and 3) energy efficiency, as explained below.

Firstly, in comparison with the manufactured-gas, the natural gas could enhance the consumer safety substantially because of the following reasons. The manufactured-gas contains substances heavier than the air, represented by propane. Accordingly, if it leaks, the manufactured-gas tends to pile from the floor, and can explode easily by catching a fire. In contrast, the natural gas has methane as a major component, which is lighter than the air. Consequently, even if it leaks, the natural gas will diffuse into the air, reducing the probability of explosion significantly. Furthermore, the flammable range of the natural gas is much smaller than that of the manufactured-gas, making the consumer safety better.

Secondly, the natural gas produces much less air pollutants, such as nitrogen oxide, sulfur oxide, and particle materials, than fossil fuel including crude oil needed to produce the manufactured-gas.

Finally, the calorific-value of the natural gas is twice as much as that of the manufactured-gas. In other words, given the same volume, the natural gas would produce energy twice as much as the manufactured-gas, reducing the transportation cost by one half.

Because of these three advantages, the natural gas has enabled one to deal with large-volume customers more easily, resulting in the shift of the core of the city gas business from the residential sector to the commercial and industrial sectors.

By taking advantage of the merits of the natural gas, Japanese city gas companies adopted a strategic change in early 1980's, and penetrated into new markets of industrial plants and commercial buildings with substantial demand of energy for manufacturing, air-conditioning and the like. Traditionally, the former market had been dominated by crude oil companies, while the latter market by electricity companies. The new strategy could be characterized by three main targets: 1) factory energy demand; 2) air-conditioning for office buildings and commercial facilities; and 3) CGS (Co-Generation System).

For the factory energy demand, city gas companies attempted to replace the crude oil by the natural gas as the source of energy for industrial furnace and boiler. At that time, the natural gas was slightly more expensive due to the additional cost for liquefaction and sea transportation with minus-temperature. However, it had the environmental advantage of emitting less air pollutants than other fossil fuels. In addition, the natural gas could be provided through underground pipes, enabling one to utilize the space for oil tanks in the factory for other purposes. Taking advantage of these merits for the customers combined with the severe sales efforts, city gas companies could increase their share in the market steadily, first around urban areas and subsequently in suburban areas.

Before 1950, the commercial air-conditioning sector in Japan was totally dominated by electricity companies. Since late 1980's, however, city gas companies has been steadily increasing their share in the market using the natural gas as the main vehicle, reaching the market share of 40% in 2005, as illustrated in Figure 2.1.2. In order to provide air-conditioning for commercial buildings by the natural gas, a gas absorption refrigerator is used, which is more competitive for large buildings and less competitive for small buildings against air-conditioning systems driven by electricity. Since the metropolitan areas tend to have large buildings, it is estimated that the market share of the city gas in these areas exceeds 50% now.

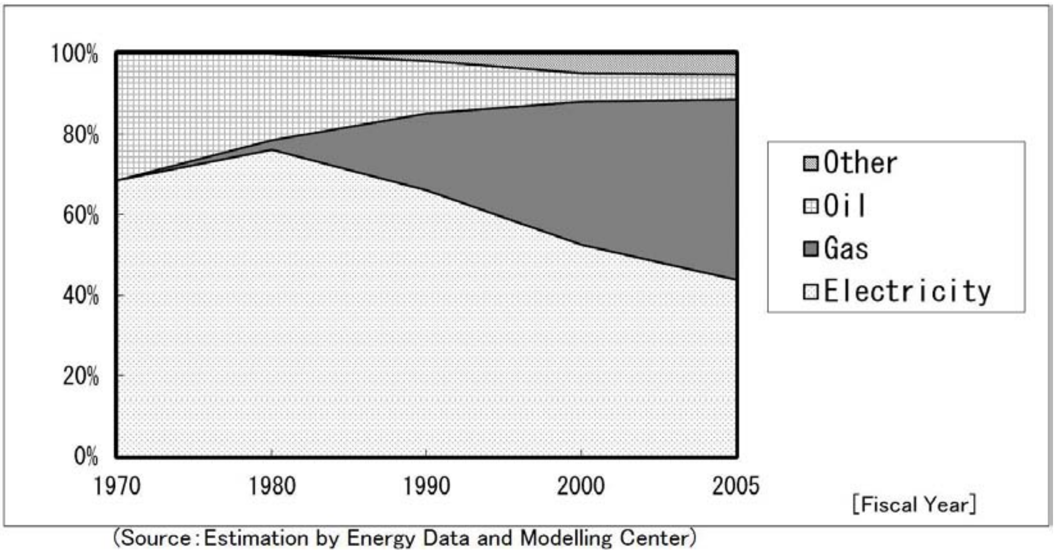

Figure 2.1.2: Market Share of Gas in the Commercial Air-Conditioning Sector in Japan

CGS is characterized by its capability to generate both electricity and heat through gas-turbine or a gas engine within the premise of a customer. It could allow the customer to save energy and utilize the heat which would be discarded at conventional power stations. Since electricity and heat can be acquired through traditional channels, e.g. electricity purchased from an electric power company and heat to be generated by a conventional boiler, the true merit of CGS has to be evaluated by comparing the cost for electricity and heat with CGS against the cost for the same amount of electricity and heat to be acquired through traditional channels. Typical customers who may enjoy the cost merit of CGS include manufacturing plants, hospitals and hotels.

The possible cost merit of CGS must be estimated quantitatively in advance with accuracy sufficient enough to convince a customer to invest in CGS. For this purpose, it is necessary to precisely estimate hourly and monthly demand of electricity and heat, which can be done often through computer simulation. This computer simulation may not be easy because a variety of sample data ought to be collected from different places of the premise of a customer. Chapter 3 deals with this problem so as to mitigate the burden of the necessary data collection.

2.2 Deregulation of the Utility Industry

Following the strategic success through penetration into new markets of industrial plants and large-scale commercial buildings, Japanese city gas companies experienced the second wave of restructuring from 1990 to 2005 due to deregulation of electricity and gas utilities.

As suppliers of a public utility, Japanese city gas companies were used to be protected by having the exclusive right to provide the gas service in their designated areas. From the government point of view, this exclusive designation policy facilitated the efficient investment in the installment of pipelines, which could be huge, by avoiding unnecessary overlaps. In exchange, the city gas companies were obligated to provide gas services at regulated prices, allowing them to obtain only reasonable profits. Typically, one designated area was isolated from others. Accordingly, there were customers outside any designated area. If such a customer was residential, the customer had to rely upon propane gas provided by a propane gas company, not a city gas company. An industrial plant outside any designated area would rely upon crude oil.

Deregulation of the gas industry has been taking place in most of developed countries in the world by now, led by the United States. In the late 1970's in the United States, a serious shortage of natural gas was caused by the regulated wellhead prices which were set much lower than market prices. While these price gaps enabled the gas utility companies to make huge profits, they offered little incentive for the natural gas producing companies to invest further in exploration of new natural gas reserves, resulting in the serious shortage of natural gas. So as to overcome this imbalance between supply and demand, the Natural Gas Policy Act (NGPA) was enacted in 1978. Since then, various market regulations have been relaxed or abolished until now. In the United Kingdom, the structural reform of British economy was promoted by the Thatcher administration so as to recover from the prolonged economic slump from 1970 to 1990. Electricity and gas industries were deregulated as a part of those measures. For the case of European Union, the deregulation of electricity and gas industries took place within a framework of establishing a single European market by removing barriers between the nations to guarantee the free movement of goods, capital, services, and people.

In parallel with this global trend, the gas utilities in Japan were liberated in March 1995. Large-scale customers with consumption of more than 2 million (m^3/year) of city gas were allowed to choose a supplier freely and price regulations were abolished for these customers. So as to motivate new entrants, the existing city gas companies were forced to provide them with transportation services at regulated prices, although new entrants were still responsible for procuring natural gas by themselves. The requirement to be a large-scale customer has been relaxed step-by-step as shown in Table 2.2.1. In April 2007, the requirement was set to have annual consumption of 100 thousand (m^3/year) or more.

In response to the deregulation, Japanese city gas companies started to introduce a new

	Mar. 1995	Nov. 1999	Apr. 2004	Apr. 2007
Scope of liberalization	2million (m3/year) or more	1million (m3/year) or more	0.5 million (m3/year) or more	0.1 million (m3/year) or more
Example of liberalized cutomers	Large-scale industrial users			
		Large-scale commercial facilities		
			Medium-size factories or large-scale hospitals	
				Small factories or business hotel
Ratio of liberalized sales volume	44%	49%	52%	59%
Gas rate for non-liberalized customers	Approval system	Notification system (only when price reduction)	Notification system (only when price reduction)	Notification system (only when price reduction)

Table 2.2.1: Deregulation of City Gas Industry in Japan

strategy, where large-scale customers outside their designated service areas would become potential targets. In this case, the competition for a city gas company would be against oil companies that could provide crude oil and other city gas companies. If the city gas company would win, it might eventually install a long-distance pipeline from its designated service area to a new customer outside its designated service area. Since this investment would be huge and time-consuming, the company would be likely to rely upon lorry transportation of the LNG in the beginning.

In order to win the new competition described above resulting from the deregulation, the key factor would be price. Accordingly, the pricing strategy is of crucial importance to the suppliers. Chapter 4 and 5 deals with this problem by developing and analyzing a mathematical model based on a game theoretic approach.

2.3 Emergence of All-Electric House Market as a Major Competitor

The industrial market and the large-scale commercial market have been liberated through deregulation and the principle of free competition now prevails. On the other hand, the residential market as well as the small-scale commercial market was not affected much by the deregulation and remained intact. However, upon entering the new century, the emergence of all-electric house market has been changing these two markets substantially.

In an all-electric house system, energy demand for cooking and hot water within household is supplied totally by electricity with IH cooking heaters and an electric "heat-pump water heating and supply system." While these technologies are not new, all-electric house systems become popular because of enhancement of price-performance of those equipment, combined with the strong promotion campaigns by electricity companies. Figure 2.3.1 and Table 2.3.1 show the dissemination of all-electric house systems in Japan. It can be observed that all-electric houses are more popular in rural areas. This is so because: 1) rural customers tend to have enough space for the hot-water cylinder, necessary to store hot water made by the heat pump system mainly at night; 2) residential customers in rural area usually use propane gas for cooking, and IH cooking is often cheaper.

	Three major metropolitan areas	Northern Japan	Other area	Total
The number of all-electric houses	1,091	225	834	2,149
The number of houses (*1)	35,613	7,477	12,800	55,890
Dissemination rate	3.1%	3.0%	6.5%	3.8%

(Source:FUKOKU MUTUAL LIFE INSURANCE COMPANY)
(*1)Number of contracts for residential use

Table 2.3.1: Dissemination of All-Electric House Systems by Area (March 2007)

The emergence of all-electric house system has been forcing city gas companies to re-

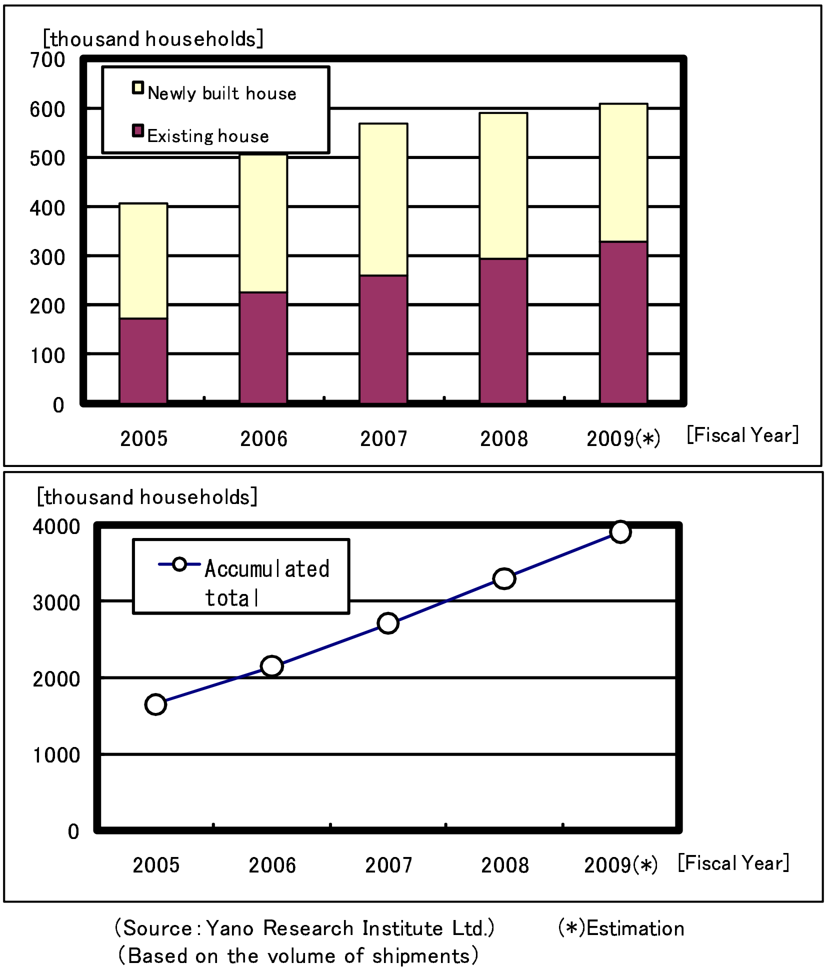

Figure 2.3.1: Dissemination of All-Electric House Systems in Japan

consider their strategy for the residential market and the small-scale commercial market. Previously, residential customers tend to choose city gas for cooking and hot water if a gas pipeline is located near their houses. When a gas pipeline is not accessible easily, they may use other energy such as propane gas or kerosene. However, if a new pipeline is installed within their vicinity, they would switch to city gas upon a chance without much cost for doing so, e.g. at the time of renovation or rebuilding. In short, if there are houses near a network of gas pipelines, the city gas company owning the network could count on the residential gas demand for a very long time. However, in the presence of all-electric house system, this presumption is no longer valid. A residential customer may choose to adopt all-electric house system upon moving into a house even with a gas pipeline, terminating the use of city gas and hence the network of gas pipelines at that house. This change of the market environment is important enough for city gas companies to reconsider their investment strategy for expanding their existing networks of gas pipelines, since it would require huge capital spending and a very long time for investment recovery.

In order to establish a new investment strategy for expanding the existing network of pipelines in face of all-electric house system, it is necessary to develop and analyze a mathematical model for capturing the cash flow of a residential area with a given network of gas pipelines, where residents move in and move out stochastically, and choose city gas or all-electric house system upon moving in. This problem is addressed in Chapter 6, where residents move in and move out according to a Markov chain in continuous time, and the relevant cash flow is described by a reward process defined on the Markov chain.

Chapter 3

Performance Analysis of Co-Generation Systems in Terms of Energy-Saving

3.1 Introduction

In management of energy demands for commercial buildings, hospitals, industrial plants and the like, a co-generation system achieves high energy-saving by simultaneously producing heat and electricity from a single source of power supply so as to satisfy the entire energy demand. Since the energy conservation is an important social requirement in Japan, which has no natural resources for energy generation, the benefits of the co-generation system have been realized by many users of the system all over Japan, as can be seen by the steady growth of the system since its introduction in 1960's.

As we mentioned in Section 2.1, however, it was not an easy task for a city gas company to disseminate co-generation systems due to the complexity involved in designing the system. Prior to the installation of a co-generation system, it is usually necessary to evaluate the impact of the system on energy-saving. In other words, one needs to define an evaluation function for describing the ratio of the total energy consumption before and after the installation of the system. In general, the total energy demand consists of the electricity demand, the cooling energy demand, the heating energy demand, and the hot water energy demand.

In order to assess the effect of a co-generation system in energy-saving, the aggregated annual sums of these energy demands, as well as their hourly and monthly consumption patterns ought to be known. Considering these stochastic elements as random variables, the total energy consumption per m^2 at a premise with or without a co-generation system can be derived as a random variable in terms of them. Let T^{CGS} and T^{Conv} be such random variables with or without a co-generation system respectively. The ultimate decision criteria for installing a co-generation system then hinges on RE defined by

$$RE = \frac{T^{Conv} - T^{CGS}}{T^{Conv}} \quad . \tag{3.1.1}$$

In general, a co-generation system is designed as a general product for a specific group of customers, e.g. apartment buildings. These customers are assumed to have a stochastically identical energy demand structure, comprising of the aggregated annual sum and the hourly and monthly consumption patterns for the electricity demand, as well as those for the cooling energy demand, the heating energy demand, and the hot water energy demand. In order to assess the level of energy-saving by a co-generation system, it is necessary to find the estimate of RE specified in (3.1.1). For this purpose, the estimates of the underlying random variables are first obtained by measuring actual heat and electricity consumptions at sampled premises. Because of the tremendous cost involved in gathering such detailed information, it is virtually impossible to collect the data from many different premises. A typical approach to overcome this difficulty is as follows.

1) Several representative premises are selected for gathering the detailed information in a complete form, using per m^2 as a unit.

2) Many more premises are chosen so as to sample only a set of aggregated statistics comprising of the stochastic components of the detailed information.

3) The detailed information obtained in 1) is transformed so that the estimates of the aggregated statistics generated from the transformed result would be the same as those obtained in 2).

4) The transformed detailed information in 3) would be used to assess the estimate of RE.

5) The assessment result of 4) is then applied to evaluate the performance of a co-generation system at a premise which is assumed to have an energy demand structure being stochastically identical to that of sampled premises.

The procedure above is common and the customization to individual premises is limited to consider their areas.

Since gathering data for 2) is still costly, it is important to understand the sensitivity of the performance of the co-generation system as a function of each of the aggregated statistics. If an aggregated statistic significantly affects the ultimate result concerning the performance of the co-generation system, the sample size for estimating this aggregated statistic may be increased so as to improve its quality. Otherwise, the sample size may be reduced or the aggregated statistic may be even discarded for cost saving. The purpose of this chapter is to develop a methodological approach for numerically exploring such sensitivities.

The structure of this chapter is as follows. In Section 3.2, a stochastic structure of various energy demands at a premise is first described. In terms of the random variables constituting the stochastic structure, the three random variables T^{Conv}, T^{CGS}, and RE, are then expressed explicitly. Corresponding to 2), five aggregated statistics are introduced in Section 3.3, and a procedure is summarized, in a succinct manner, for transforming the detailed information gathered in 1) into the modified detailed information discussed in 4). In Sections 3.4, the sensitivity of the performance of the co-generation system as a function of each of the aggregated statistics is explored numerically based on actual data involving apartment buildings, hotels and office buildings.

3.2 Evaluation of the Effect of a Co-generation System for Energy-Saving

In order to evaluate the effect of a co-generation system for energy-saving, we first describe a stochastic structure of the energy demand at a premise. In this regard, it is important to keep track of monthly and hourly demand patterns because of the seasonality and the daily life cycle. However, the fluctuation of the energy usage over different days may not be so essential. Keeping this observation in mind, we define the following random variables. For notational convenience we define $\mathcal{N}_k = \{1, 2, \cdots, k\}$.

$$\underline{\underline{E}} = [E_{ij}] \ ; \ E_{ij} \quad : \quad \text{the electricity demand per m}^2 \text{ at a premise for the month } i \in \mathcal{N}_{12} \text{ and the hour } j \in \mathcal{N}_{24} \tag{3.2.1}$$

$$\underline{\underline{C}} = [C_{ij}] \ ; \ C_{ij} \quad : \quad \text{the cooling energy demand per m}^2 \text{ at a premise for the month } i \in \mathcal{N}_{12} \text{ and the hour } j \in \mathcal{N}_{24} \tag{3.2.2}$$

$$\underline{\underline{H}} = [H_{ij}] \ ; \ H_{ij} \quad : \quad \text{the heating energy demand per m}^2 \text{ at a premise for the month } i \in \mathcal{N}_{12} \text{ and the hour } j \in \mathcal{N}_{24} \tag{3.2.3}$$

$$\underline{\underline{W}} = [W_{ij}] \ ; \ W_{ij} \quad : \quad \text{the hot water energy demand per m}^2 \text{ at a premise for the month } i \in \mathcal{N}_{12} \text{ and the hour } j \in \mathcal{N}_{24} \tag{3.2.4}$$

The stochastic nature of the total energy consumption per year per m^2 is characterized by

$$\underline{\underline{Y}} = [\underline{\underline{E}}, \underline{\underline{C}}, \underline{\underline{H}}, \underline{\underline{W}}] \quad . \tag{3.2.5}$$

The performance measure of a co-generation system, given as RE in (3.1.1), can then be described as

$$RE = F(\underline{\underline{Y}}) \quad , \tag{3.2.6}$$

where F denotes the evaluation function. In what follows, a procedure to construct F is described explicitly. The following notation is employed.

m_i : the number of days in the month $i \in \mathcal{N}_{12}$ (3.2.7)

η_E^* : the average power generation efficiency across all thermal power stations in Japan, i.e. the generated power in jule/the consumed fuel in jule (3.2.8)

M_G : the capacity in jule of the gas engine generator at the premise (3.2.9)

$L_{ij} \stackrel{\text{def}}{=} \min\left\{\frac{E_{ij}}{M_G}, 1\right\}$: the work load of the gas engine for the month i and the hour j (3.2.10)

L_{min} : the minimum work load of the gas engine (3.2.11)

$\eta_E(L_{ij})$: the power generation efficiency of the gas engine generator for the month i and the hour j given L_{ij}, where the underlying function structure is determined empirically (3.2.12)

λ_C : the energy efficiency of the cooling system (3.2.13)

λ_H : the energy efficiency of the heating system (3.2.14)

λ_W : the energy efficiency of the hot water system (3.2.15)

$\eta_H(L_{ij})$: the heat recovery efficiency of the gas engine generator (3.2.16)

S_{ij} : the cumulative amount of energy saved in a form of hot water at the beginning of the hour j in the month i (3.2.17)

M_S : the capacity for accumulating the energy in a form of hot water (3.2.18)

L : the ratio of energy loss per hour while saved in a form of hot water (3.2.19)

The basic values of those parameters are shown in Table 3.2.1.

Table 3.2.1: Basic Value of the Underlying Parameters

η_E^*	0.35
λ_C	2.00
λ_H	0.80
λ_W	0.80
M_S	$5M_G$
L_{min}	0.3
L	0.05

L_{ij}	*0.50*	*0.75*	*1.00*
η_E	*0.30*	*0.33*	*0.35*
η_H	*0.42*	*0.41*	*0.40*

We begin the construction of F with specification of the random variable T^{conv} denoting the total energy consumption per year per m^2 at a premise without a co-generation system. One sees that

$$T^{conv} = \sum_{i \in \mathcal{N}_{12}} m_i \sum_{j \in \mathcal{N}_{24}} \left(\frac{E_{ij}}{\eta_E^*} + \frac{C_{ij}}{\lambda_C} + \frac{H_{ij}}{\lambda_H} + \frac{W_{ij}}{\lambda_W} \right) \quad . \tag{3.2.20}$$

It may be worth noting that, by convention, the actual demand of one source is inflated into the necessary energy consumption at the input level by dividing it by the corresponding efficiency.

We next turn our attention to derive T^{CGS} denoting the total energy consumption per year per m^2 at the premise with a co-generation system. Let G_{ij}^E be the total energy consumption of the gas engine generator for the hour j in the month i used to satisfy the electricity demand. One then sees that

$$G_{ij}^E = \frac{\min\{E_{ij}, M_G\}}{\eta_E(L_{ij})} \text{ if } L_{ij} \geq L_{min} \text{ else } G_{ij}^E = 0 \quad . \tag{3.2.21}$$

If $E_{ij} > M_G$ or $E_{ij} < L_{min} M_G$, one faces the shortage which should be filled by purchasing E_{ij}^B from an external electric power company, where

$$E_{ij}^B = (E_{ij} - M_G)^+ \quad . \tag{3.2.22}$$

Here $(x)^+ = x$ if $x > 0$ and zero else. For the hot water system, the energy recovered from the gas engine power generator through the co-generation system can be utilized. More specifically, the total energy consumption for the hour j in the month i used to satisfy the hot water demand, denoted by G_{ij}^W is given by

$$G_{ij}^W = \left(\frac{W_{ij}}{\lambda_W} - A_{ij}\right)^+ ; A_{ij} = \eta_H(L_{ij})G_{ij}^E + S_{ij} \quad , \tag{3.2.23}$$

where A_{ij} denotes the sum of energy recovered from G_{ij}^E and energy saved in a form of hot water. If A_{ij} could cover the hot water consumption W_{ij}/λ_W, the hot water cost is free. Otherwise, the hot water consumption can be reduced by A_{ij}.

The total energy consumption for heating denoted by G_{ij}^H can be treated in a similar manner, except that the priority is given to hot water for using the recovered and saved energy A_{ij}. One has

$$G_{ij}^H = \left(\frac{H_{ij}}{\lambda_H} - \hat{A}_{ij}\right)^+ ; \hat{A}_{ij} \stackrel{\text{def}}{=} \left(A_{ij} - \frac{W_{ij}}{\lambda_W}\right)^+ \quad . \tag{3.2.24}$$

Since the lowest priority is given to the cooling system concerning the usage of the recovered energy, the total energy consumption for cooling, denoted by G_{ij}^C is given by

$$G_{ij}^C = \left(\frac{C_{ij}}{\lambda_C} - \hat{\hat{A}}_{ij}\right)^+ ; \hat{\hat{A}}_{ij} = \left(\hat{A}_{ij} - \frac{H_{ij}}{\lambda_H}\right)^+ \quad . \tag{3.2.25}$$

The cummulative heat at the beginning of the hour $j+1$ is then updated to

$$S_{i,\ j+1} = \min\left\{\left(\hat{\hat{A}}_{ij} - \frac{C_{ij}}{\lambda_C}\right)^+, M_S\right\}(1-L) \quad . \tag{3.2.26}$$

Finally, the total energy consumption per year per m^2 at the premise with the co-generation system is given by

$$T^{CGS} = \sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\left(\frac{E_{ij}^B}{\eta_E^*} + G_{ij}^C + G_{ij}^H + G_{ij}^W\right) \quad . \tag{3.2.27}$$

Remark 3.2.1 *In some co-generation systems, the recovered energy cannot be used for cooling system. In such a case, Equations (3.2.25) and (3.2.26) are replaced by* $G^C_{ij} = \frac{C_{ij}}{\lambda_C}$ *and* $S_{i,\ j+1} = \min\left\{\left(\hat{\hat{A}}_{ij}\right)^+, M_S\right\}(1-L)$ *respectively.*

Finally, the evaluation function F can be constructed by measuring the relative enhancement for energy-saving due to the co-generation system, that is,

$$RE = F(\underline{\underline{Y}}) = \frac{T^{Conv} - T^{CGS}}{T^{Conv}} \quad . \tag{3.2.28}$$

3.3 Aggregated Statistics and Transformation of Detailed Information

In order to implement 2) in Section 3.1, we introduce the following five aggregated statistics X_1 through X_5.

$$X_1 = \sum_{i\in\mathcal{N}_{12}} \sum_{j\in\mathcal{N}_{24}} (E_{ij} + W_{ij} + H_{ij} + C_{ij}) \tag{3.3.1}$$

$$X_2 = \frac{\sum_{i\in\mathcal{N}_{12}} \sum_{j\in\mathcal{N}_{24}} (W_{ij} + H_{ij} + C_{ij})}{\sum_{i\in\mathcal{N}_{12}} \sum_{j\in\mathcal{N}_{24}} E_{ij}} , \tag{3.3.2}$$

$$X_3 = \frac{\sum_{i\in\mathcal{N}_{12}} \sum_{j\in\mathcal{N}_{24}} (H_{ij} + C_{ij})}{\sum_{i\in\mathcal{N}_{12}} \sum_{j\in\mathcal{N}_{24}} W_{ij}} , \tag{3.3.3}$$

$$\begin{aligned} X_4 = {} & \frac{(\sum_{i\in\mathcal{N}_{12}} \sum_{j\in\mathcal{N}_{24}} E_{ij})/12}{\max_i\{\sum_{j\in\mathcal{N}_{24}} E_{ij}\}} + \frac{(\sum_{i\in\mathcal{N}_{12}} \sum_{j\in\mathcal{N}_{24}} W_{ij})/12}{\max_i\{\sum_{j\in\mathcal{N}_{24}} W_{ij}\}} \\ & + \frac{(\sum_{i\in\mathcal{N}_{12}} \sum_{j\in\mathcal{N}_{24}} H_{ij})/12}{\max_i\{\sum_{j\in\mathcal{N}_{24}} H_{ij}\}} + \frac{(\sum_{i\in\mathcal{N}_{12}} \sum_{j\in\mathcal{N}_{24}} C_{ij})/12}{\max_i\{\sum_{j\in\mathcal{N}_{24}} C_{ij}\}} \end{aligned} \tag{3.3.4}$$

$$\begin{aligned} X_5 = {} & \sum_{i\in\mathcal{N}_{12}} \left\{ \frac{(\sum_{j\in\mathcal{N}_{24}} E_{ij})/24}{\max_j E_{ij}} + \frac{(\sum_{j\in\mathcal{N}_{24}} W_{ij})/24}{\max_j W_{ij}} \right. \\ & \left. + \frac{(\sum_{j\in\mathcal{N}_{24}} H_{ij})/24}{\max_j H_{ij}} + \frac{(\sum_{j\in\mathcal{N}_{24}} C_{ij})/24}{\max_j C_{ij}} \right\} \end{aligned} \tag{3.3.5}$$

We note that X_1 describes the annual energy demand, while X_2 is the annual heat-to-electricity ratio, that is, the ratio of the annual heat demand to the annual electricity demand. Similarly, X_3 is the annual cooling and heating-to-hot water ratio. Somewhat complicated is X_4 expressing the yearly load factor, i.e. the ratio of the monthly average demand to the monthly peak-demand. X_5 denotes the daily load factor in a similar manner, that is, the ratio of the hourly average demand to the hourly peak-demand.

We now discuss 3) of Section 3.1. Assuming that the five statistics are sampled from many premises, an algorithmic procedure is given to transform the detailed information collected from only a few premises so that the transformed detailed information would produce the estimates of the five statistics as sampled from many premises. More formally, let $\underline{\underline{y}} = [\underline{\underline{e}}, \underline{\underline{c}}, \underline{\underline{h}}, \underline{\underline{w}}]$ be the estimate of $\underline{\underline{Y}} = [\underline{\underline{E}}, \underline{\underline{C}}, \underline{\underline{H}}, \underline{\underline{W}}]$ collected from a few premises. Similarly, corresponding to $\underline{X} = [X_1, X_2, \cdots, X_5]$, we define $\underline{x} = [x_1, x_2, \cdots, x_5]$ for the five statistics sampled from many more premises. Our purpose is to find the transformation of $\underline{\underline{y}} = [\underline{\underline{e}}, \underline{\underline{c}}, \underline{\underline{h}}, \underline{\underline{w}}]$ into $\hat{\underline{\underline{y}}} = [\hat{\underline{\underline{e}}}, \hat{\underline{\underline{c}}}, \hat{\underline{\underline{h}}}, \hat{\underline{\underline{w}}}]$ so that $\hat{\underline{x}} = [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_5]$, derived from $\hat{\underline{\underline{y}}}$ based on Equations (3.3.1) through (3.3.5), is equal to $\underline{x} = [x_1, x_2, \cdots, x_5]$.

The transformation is constructed in five stages, where the k-th stage establishes $\hat{x}_k = x_k$, while preserving $\hat{x}_r = x_r$ for $r = 1, \cdots, k-1$. Let $\underline{x}(\underline{\underline{z}}) = [x_1(\underline{\underline{z}}), x_2(\underline{\underline{z}}), \cdots, x_5(\underline{\underline{z}})]$ be obtained from $\underline{\underline{z}}$ based on Equations (3.3.1) through (3.3.5). For $k = 1$, we define

$$\hat{\underline{\underline{y}}}(1) \leftarrow \alpha_1 \times \underline{\underline{y}}; \quad \alpha_1 = \frac{x_1}{x_1(\underline{\underline{y}})} \quad . \tag{3.3.6}$$

It is clear that $x_1(\hat{\underline{\underline{y}}}(1)) = x_1$. For $k = 2$, we introduce

$$\hat{\underline{\underline{e}}}(2) \leftarrow \alpha_2 \times \hat{\underline{\underline{e}}}(1) \ ; \quad \hat{\underline{\underline{c}}}(2) \leftarrow \beta_2 \times \hat{\underline{\underline{c}}}(1) \ ; \quad \hat{\underline{\underline{h}}}(2) \leftarrow \beta_2 \times \hat{\underline{\underline{h}}}(1) \ ; \quad \hat{\underline{\underline{w}}}(2) \leftarrow \beta_2 \times \hat{\underline{\underline{w}}}(1) \ , \tag{3.3.7}$$

where

$$\alpha_2 = \frac{x_1}{(1 + x_2) \sum_{i \in \mathcal{N}_{12}} \sum_{j \in \mathcal{N}_{24}} \hat{e}_{ij}(1)} \tag{3.3.8}$$

and

$$\beta_2 = \frac{x_1 x_2}{(1+x_2)\sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}(\hat{c}_{ij}(1)+\hat{h}_{ij}(1)+\hat{w}_{ij}(1))} \quad , \tag{3.3.9}$$

yielding $\underline{\underline{\hat{y}}}(2)$. After a little algebra, one finds that $x_2(\underline{\underline{\hat{y}}}(2)) = x_2$ and $x_1(\underline{\underline{\hat{y}}}(2)) = x_1$.

Similarly for $k = 3$, the transformation stage is constructed by

$$\underline{\underline{\hat{e}}}(3) \leftarrow \underline{\underline{\hat{e}}}(2) \ ; \quad \underline{\underline{\hat{c}}}(3) \leftarrow \beta_3 \times \underline{\underline{\hat{c}}}(2) \ ; \quad \underline{\underline{\hat{h}}}(3) \leftarrow \beta_3 \times \underline{\underline{\hat{h}}}(2) \ ; \quad \underline{\underline{\hat{w}}}(3) \leftarrow \alpha_3 \times \underline{\underline{\hat{w}}}(2) \ , \tag{3.3.10}$$

where

$$\alpha_3 = \frac{x_1 - \sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(2)}{(1+x_3)\sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\hat{w}_{ij}(2)} \quad , \tag{3.3.11}$$

and

$$\beta_3 = \frac{x_3\{x_1 - \sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(2)\}}{(1+x_3)\sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\{\hat{c}_{ij}(2)+\hat{h}_{ij}(2)\}} \quad , \tag{3.3.12}$$

resulting in $\underline{\underline{\hat{y}}}(3)$. One sees that $x_3(\underline{\underline{\hat{y}}}(3)) = x_3$, $x_2(\underline{\underline{\hat{y}}}(3)) = x_2$ and $x_1(\underline{\underline{\hat{y}}}(3)) = x_1$.

The final two stages are somewhat more complicated. For $k = 4$, we define

$$\hat{e}_{ij}(4) \leftarrow \frac{\hat{e}_{ij}(3)}{\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(3)}\left\{\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(3) + \alpha_4^e\left(\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(3) - \frac{1}{12}\sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(3)\right)\right\} \tag{3.3.13}$$

for all $i \in \mathcal{N}_{12}$ and $j \in \mathcal{N}_{24}$. Here, α_4^e and β_4^e are defined by

$$\alpha_4^e \stackrel{\text{def}}{=} -\frac{\max_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(3) - \dfrac{1}{x_4\beta_4^e}(\sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(3))/12}{\max_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(3) - \frac{1}{12}\sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(3)} \tag{3.3.14}$$

and

$$\beta_4^e \stackrel{\text{def}}{=} \frac{1}{x_4(\underline{\underline{\hat{y}}}(3))}\ \frac{(\sum_{i\in\mathcal{N}_{12}}\sum_{j\in\mathcal{N}_{24}}\hat{y}_{ij}(3))/12}{\max_{i\in\mathcal{N}_{12}}\{\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(3)\}} \quad . \tag{3.3.15}$$

$\hat{c}_{ij}(4), \hat{h}_{ij}(4)$ and $\hat{w}_{ij}(4)$ can be defined similarly based on (3.3.13) through (3.3.15), where $\hat{e}_{ij}(3)$ are replaced by $\hat{c}_{ij}(3)$, $\hat{h}_{ij}(3)$, and $\hat{w}_{ij}(3)$, respectively, leading to $\underline{\underline{\hat{y}}}(4)$. It can be confirmed that $x_4(\underline{\underline{\hat{y}}}(4)) = x_4$, $x_3(\underline{\underline{\hat{y}}}(4)) = x_3$, $x_2(\underline{\underline{\hat{y}}}(4)) = x_2$ and $x_1(\underline{\underline{\hat{y}}}(4)) = x_1$.

Finally for $k = 5$, we define

$$\hat{e}_{ij}(5) \leftarrow \hat{e}_{ij}(4) + \alpha^e_{5:i}\left(\hat{e}_{ij}(4) - \frac{1}{24}\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(4)\right) \quad , \tag{3.3.16}$$

where $\alpha^e_{5:i}$ and $\beta^e_{5:i}$ are given for $i \in \mathcal{N}_{12}$ by

$$\alpha^e_{5:i} \stackrel{\text{def}}{=} -\frac{\max_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(4) - \dfrac{1}{x_5\beta^e_{5:i}}(\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(4))/24}{\max_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(4) - \frac{1}{24}\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(4)} \tag{3.3.17}$$

and

$$\beta^e_{5:i} \stackrel{\text{def}}{=} \frac{1}{x_5(\underline{\underline{\hat{y}}}(4))}\frac{(\sum_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(4))/24}{\max_{j\in\mathcal{N}_{24}}\hat{e}_{ij}(4)} \quad \text{for } i \in \mathcal{N}_{12} \quad . \tag{3.3.18}$$

As for the case of $k = 4$, $\hat{c}_{ij}(5), \hat{h}_{ij}(5)$ and $\hat{w}_{ij}(5)$ can be obtained from (3.3.16) through (3.3.18), where $\hat{e}_{ij}(4)$ are replaced by $\hat{c}_{ij}(4)$, $\hat{h}_{ij}(4)$, and $\hat{w}_{ij}(4)$, respectively, yielding $\underline{\underline{\hat{y}}}(5)$. It follows that $x_5(\underline{\underline{\hat{y}}}(5)) = x_5$, $x_4(\underline{\underline{\hat{y}}}(5)) = x_4$, $x_3(\underline{\underline{\hat{y}}}(5)) = x_3$, $x_2(\underline{\underline{\hat{y}}}(5)) = x_2$ and $x_1(\underline{\underline{\hat{y}}}(5)) = x_1$, completing the transformation stated in 3) of Section 3.1.

Through the procedure above, RE in (3.2.28) can be rewritten as

$$\hat{RE} = F(\underline{\underline{\hat{Y}}}) \quad , \tag{3.3.19}$$

which can be evaluated numerically given the estimate $\underline{\underline{\hat{y}}}$ of $\underline{\underline{Y}}$.

3.4 Numerical Examples for Exploring the Performance Sensitivity of the Co-Generation System

In the previous sections, we have discussed in detail a systematic approach for evaluating the performance of the co-generation system along the line of the procedure 1) through 5) given in Section 3.1. The key consideration in the approach is the balance between the information gathering cost and the reliability of the estimated performance. Since the five aggregated statistics introduced in Section 3.3 represent the information gathered from

Table 3.4.1: Standard Deviations of Five Aggregated Statistics

	X_1	X_2	X_3	X_4	X_5
Apartment Buildings	38	53	31	10	30
Hotels	26	7	14	10	30
Offices	19	26	73	10	30

large data samples, it is important to understand how sensitive the performance of the co-generation system is as a function of each of the five aggregated statistics. For this purpose, we treat $\hat{RE}$ in (3.3.19) as a function of $\underline{X} = [X_1, \cdots, X_5]$ and rewrite $\hat{RE}$ as

$$\hat{RE}(\underline{X}) = F(\underline{\hat{Y}}) \quad . \tag{3.4.1}$$

The purpose of this section is to explore the sensitivity of $\hat{RE}(\underline{X})$ with respect to X_i, $i = 1, \cdots, 5$, based on actual data involving apartment buildings, hotels and office buildings.

For the detailed information in 1) of Section 3.1, the sample size of three is taken for each of the three categories above. In order to conduct the sensitivity analysis numerically, we define $\underline{\mu} = [\mu_1, \cdots, \mu_5]$ and $\underline{\sigma} = [\sigma_1, \cdots, \sigma_5]$, where $\mu_i = E[X_i]$ and $\sigma_i^2 = Var[X_i]$, $i = 1, \cdots, 5$. Of interest is then

$$Sen(i) = |\hat{RE}(\underline{\mu} + \sigma_i \underline{1}_i) - \hat{RE}(\underline{\mu} - \sigma_i \underline{1}_i)| \quad , \tag{3.4.2}$$

where $\underline{1}_i$ is the vector with all components being 0 except the i-th component of 1.

For $\underline{\mu}$, we count on the detailed information. The standard deviation vector $\underline{\sigma}$ is obtained from the literature, see e.g. [2], [9], [12], [13]. Since the collected detailed information is proprietary, $\underline{\mu}$ cannot be disclosed. So as to provide some sense concerning $\underline{\sigma}$ for the three categories above, we express σ_i in terms of the percentage of μ_i, $i = 1, \cdots, 5$, as given in Table 3.4.1. The ultimate measure for the sensitivity analysis is defined as

$$\rho(i) = \frac{Sen(i)}{\sum_{i=1}^{5} Sen(i)} \quad , i = 1 \cdots, 5 \quad , \tag{3.4.3}$$

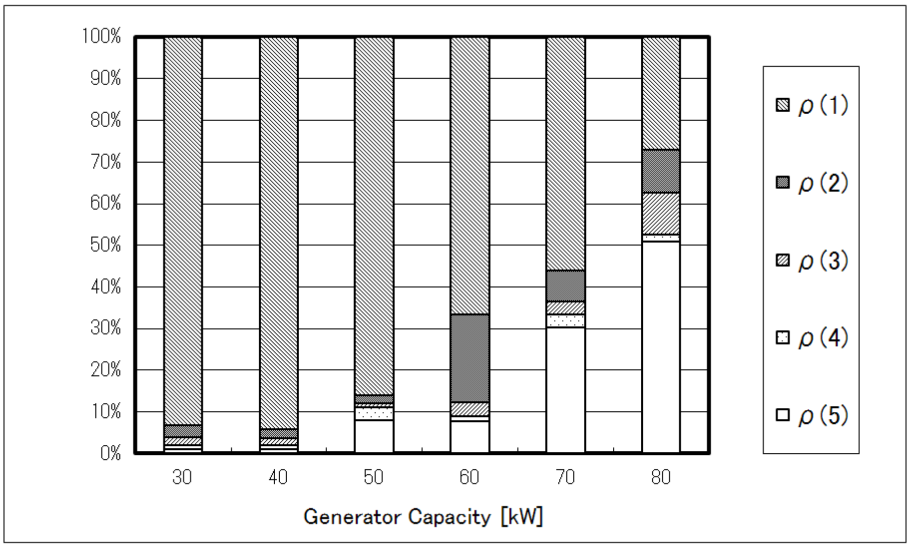

Figure 3.4.1: Sensitivity Vector ρ for Apartment Buildings

which can be disclosed explicitly. For apartment buildings, X_1 affects the performance of the co-generation system most, while X_5 for hotels and office buildings, as we will see.

3.4.1 Numerical Results for Apartment Buildings

Figure 3.4.1 shows sensitivity vector $\underline{\rho} = [\rho(1), \cdots, \rho(5)]$ for the category of apartment buildings, where the generator capacity of the co-generation system is varied from 30 kW to 80 kW. One observes that the annual energy demand X_1 affects the performance of the co-generation system most when the capacity of the generator is between 30 kW and 50 kW. From an economic point of view, it is quite reasonable to install a generator of this size, since it works as the base-load generator and the investment recovery period becomes shorter. Accordingly, one should focus on estimating X_1 most precisely.

This result may be explained as follows. If the capacity of the generator is relatively small, the electricity generated by the co-generation system is likely to be submerged in the demand-fluctuation curve of a day, as demonstrated in Figure 3.4.2. Under this condition, the generator operates at full capacity and the hourly change of the energy demand in a day

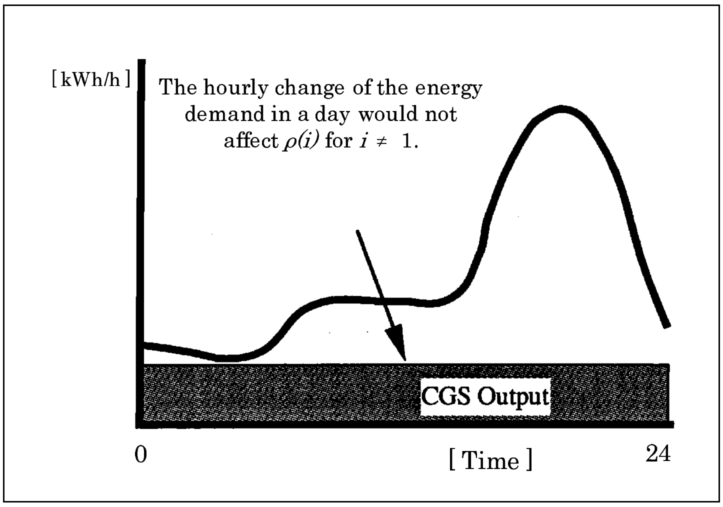

Figure 3.4.2: Typical Daily Energy Demand Pattern for Apartment Buildings

would not affect $\rho(i)$ for $i \neq 1$ at all.

3.4.2 Numerical Results for Hotels

Figure 3.4.3 shows $\underline{\rho} = [\rho(1), \cdots, \rho(5)]$ for the categroy of hotels, where the generator capacity is varied from 200 kW to 700 kW. One observes that the daily load factor X_5 affects the performance of the co-generation system most when the capacity of the generator is between 400 kW and 600kW. According to Figure 3.4.4, within this range of the generator capacity, the energy-saving is relatively high, with maximum efficiency between 500 kW and 600 kW. The co-generation system works quite effectively since the demand and the supply of heat and electricity are well balanced. For hotels, within this range of the capacity of the co-generation system, one may conclude that the daily load factor X_5 has the greatest influence on the performance among the five statistics.

3.4.3 Numerical Results for Office Buildings

For the case of office buildings, X_5 also has the biggest impact on the performance of co-generation system among the five statistics, as can be seen in Figure 3.4.5. The energy demand of the office building during the night is very low and the co-generation system usually works only during the standard working hours as shown in Figure 3.4.6. The value

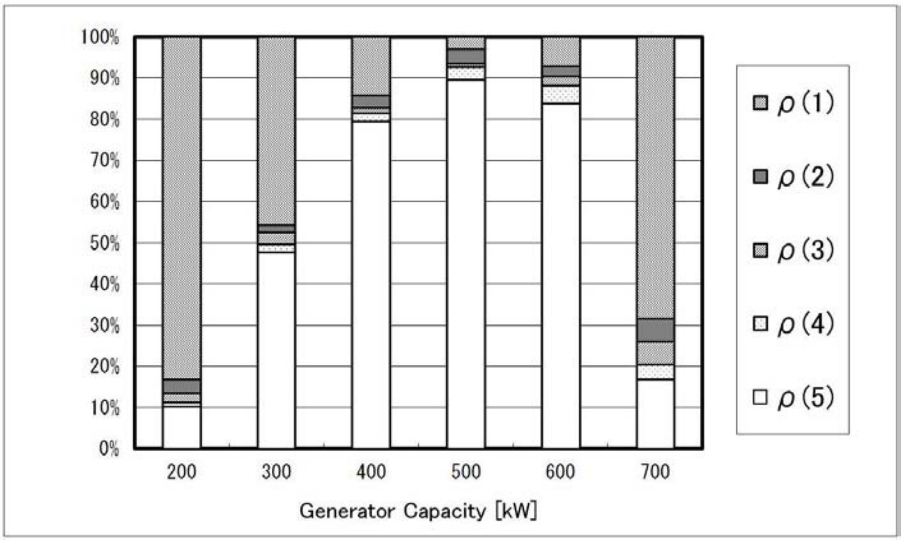

Figure 3.4.3: Sensitivity Vector ρ for Hotels

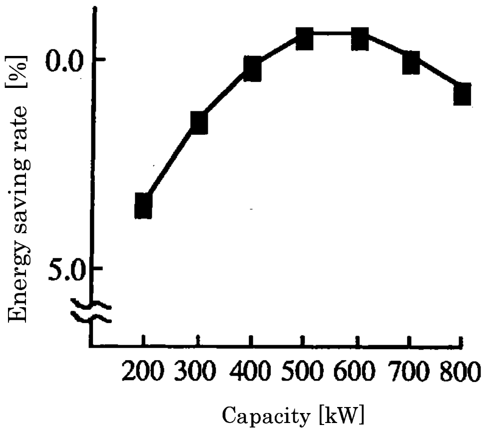

Figure 3.4.4: The Energy-Saving Versus the Capacity of the Generator for Hotels

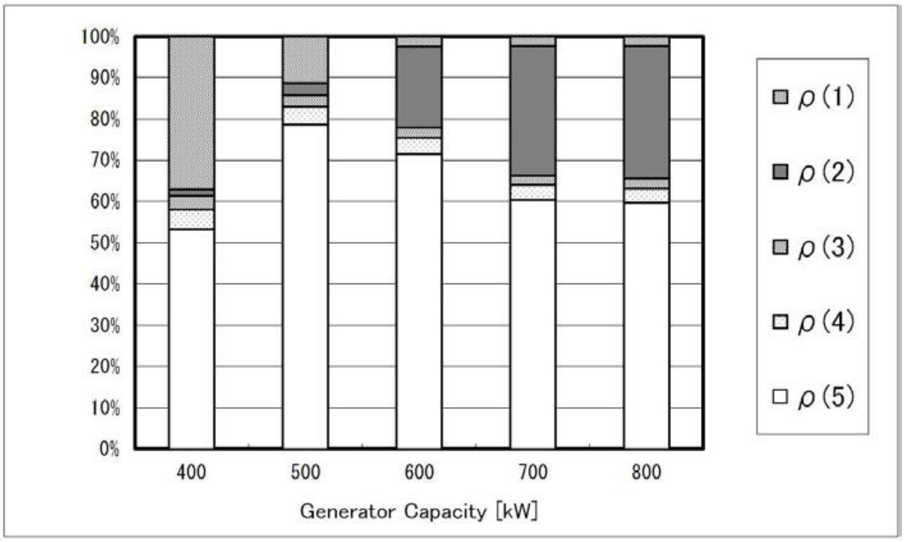

Figure 3.4.5: Sensitivity Vector ρ for Office Buildings

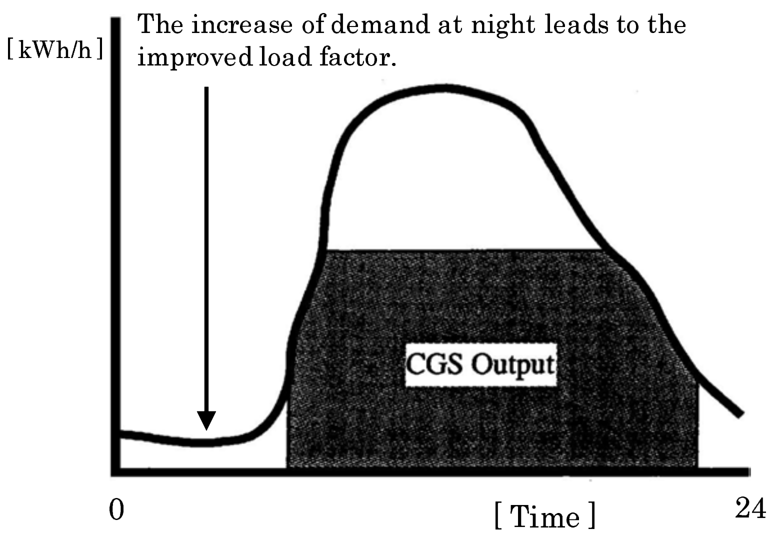

Figure 3.4.6: Typical Energy Demand Pattern for Office Buildings

of X_5 is directly affected by the energy demand at night. As the energy demand at night increases, the operating time of the co-generation system becomes longer, which leads to improvement of the performance.

Chapter 4

On Non-Existence of Nash Equilibrium of M Person Game with Pure Strategy for Delivery Services

4.1 Introduction

Competitive market models for homogeneous products and services such as the energy supply can be traced back to 1920's. The pioneering paper by Hotelling [7] develops a duopoly model where customers are distributed uniformly over a finite line and serviced by two suppliers who choose their locations and prices so as to maximize their profits. If the two suppliers are not located relatively far apart, it is shown by D'Aspremont et al [3] that Nash equilibrium does not exist. Subsequently, the Hotelling model has been extended in several directions. Economides [4] deals with the case where customers are distributed uniformly on a bounded plane. Anderson [1] incorporates stackelberg leadership within the context of the Hotelling model. Other variations include Thisse and Vives [17], Zhang and Teraoka [18] and Rath [19]. Gabszewicz and Thisse [5] provide an excellent review of the literature. More recently, for a spatially duopoly model with customers located at different nodes having separate demand functions, Matsubayashi et al. [11] establish a necessary and sufficient condition for the existence of Nash equilibrium and develop computational algorithms for finding the equilibrium point. When mixed strategies are allowed, Takahashi and Sumita [16] derive two types of Nash equilibriums explicitly for a two person model.

The purpose of this chapter is to develop an M person game with pure strategy, describing a competitive market of a homogeneous service such as LNG lorry transportation. The market consists of M suppliers and N customers, where each supplier offers a uniform price upon delivery to all customers. Locations of suppliers and customers are fixed. The thrust of this chapter is to show that, except under a rather peculiar necessary and sufficient condition, Nash equilibrium for pure strategy does not exist, demonstrating that the suppliers exercise their price strategies in a cyclic manner indefinitely. The structure of this chapter is as follows. In Seciton 4.2, a competitive market model is formally introduced and the game-theoretic framework is established. A necessary and sufficient condition is derived in Section 4.3 for existence of Nash equilibrium with pure strategy. It is shown that the Nash equilibrium is unique, if any, and rather peculiar in that all suppliers adopt the price upper bound U. Finally in Section 4.4, a duopoly model is discussed explicitly demonstrating the cyclic phenomenon of the suppliers in exercising their price strategies so as to maximize their profits.

4.2 Model Description

We consider a market consisting of M suppliers and N customers as depicted in Figure 4.2.1, where each supplier provides a homogeneous service such as delivering propane gas cylinders. Each customer may represent one large industry or a group of residents in the same district. Let $\mathcal{M} = \{1, 2, 3, \cdots, M\}$ and $\mathcal{N} = \{1, 2, 3, \cdots, N\}$ be the set of suppliers and the set of customers respectively. The cost for supplier $i \in \mathcal{M}$ to provide a unit of service to customer $j \in \mathcal{N}$ is denoted by c_{ij}.

Since the service under consideration is typically an energy supply service, it is natural

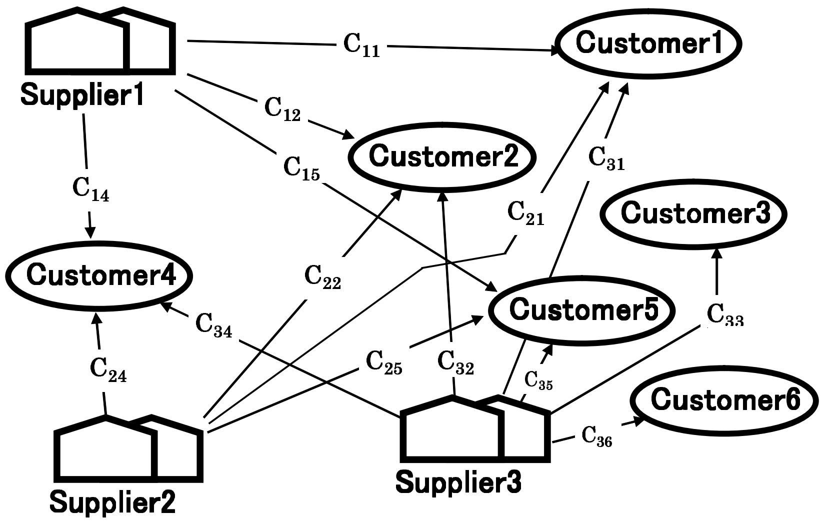

Figure 4.2.1: M Supplier-N Customer Model with $M = 3$ and $N = 6$

to assume that there exists a price upper bound U. In our model, each supplier has to offer a uniform price upon delivery to all customers, and this uniform price is denoted by $\pi_i, i \in \mathcal{M}$. Supplier i may offer the service to customer j only when it results in a positive return to do so. In other words, supplier i may offer the service to customer j only if $c_{ij} < \pi_i$. In order to avoid trivial cases, we assume that each supplier can offer the service to at least one of the customers so that

$$\min_{j \in \mathcal{N}} c_{ij} \stackrel{\text{def}}{=} c_i < \pi_i \leq U \quad \text{for all} \quad i \in \mathcal{M} \quad . \tag{4.2.1}$$

Let D_j be the total demand of customer j . We assume that the production capacity of each supplier is large enough to cover the entire demand $\sum_{j \in \mathcal{N}} D_j$. If there exists only one supplier who offers the lowest price to customer j , the supplier monopolizes the demand of customer j . Should k different suppliers offer the same lowest price to customer j , then each of such suppliers would sell D_j/k to customer j . In what follows, we describe an M person game defined on the strategy set S where

$$S = \prod_{i=1}^{M} S_i \ ; \ S_i = [c_i, U] i \in \mathcal{M} \quad .$$

Given $\underline{\pi}^T \stackrel{\text{def}}{=} [\pi_1, \pi_2, \cdots, \pi_M] \in S$, let $P_i(\underline{\pi})$ be the payoff function of supplier i. In order to define the function specifically, the following index sets are introduced. Given $\underline{\pi}^T \in S$, we define for $j \in \mathcal{N}$ the set of suppliers available to provide service to customer j by

$$AV_j(\underline{\pi}) = \{m \in \mathcal{M} | \pi_m > c_{mj}\} \qquad . \tag{4.2.2}$$

We also define for $i \in \mathcal{M}$,

$$LE_i(\underline{\pi}) = \{m \in \mathcal{M} | \pi_i > \pi_m\} \qquad ; \tag{4.2.3}$$

$$LA_i(\underline{\pi}) = \{m \in \mathcal{M} | \pi_i < \pi_m\} \qquad ;$$

$$EQ_i(\underline{\pi}) = \{m \in \mathcal{M} | \pi_i = \pi_m\} \qquad . \tag{4.2.4}$$

It should be noted that $AV_j(\underline{\pi})$ consists of those suppliers who can offer the service to customer j because a positive return results from doing so, and $LE_i(\underline{\pi})$ is the set of those suppliers who would eliminate supplier i if they happen to offer the service to the same customer. Similarly $LA_i(\underline{\pi})$ consists of those suppliers who would be eliminated by supplier i. With those suppliers in $EQ_i(\underline{\pi})$, supplier i would split the demand equally, should they offer the lowest price to the same customer simultaneously.

Let $W_{ij}(\underline{\pi})$ be the set of suppliers who would offer the service to customer j together with supplier i . Using the above notation, $W_{ij}(\underline{\pi})$ can be written as

$$W_{ij}(\underline{\pi}) = \begin{cases} \{m \in \mathcal{M} | m \in EQ_i(\underline{\pi}) \cap AV_j(\underline{\pi})\} & \text{if} \quad AV_j(\underline{\pi}) \cap LE_i(\underline{\pi}) = \emptyset \text{ and } i \in AV_j(\underline{\pi}) \\ \emptyset & \text{if} \quad AV_j(\underline{\pi}) \cap LE_i(\underline{\pi}) \neq \emptyset \text{ or } i \notin AV_j(\underline{\pi}) \end{cases} \tag{4.2.5}$$

It should be noted that $W_{ij}(\underline{\pi}) = \emptyset$ if either supplier i cannot gain positive profit by offering service to customer j so that $i \notin AV_j(\underline{\pi})$, or supplier i does not offer the lowest price to customer j. In the latter case, there exists $m' \in \mathcal{M}$ satisfying $m' \in LE_i(\underline{\pi})$ and $m' \in AV_j(\underline{\pi})$, and hence $AV_j(\underline{\pi}) \cap LE_i(\underline{\pi}) \neq \emptyset$. When supplier i offer the lowest price to customer j, one sees that $AV_j(\underline{\pi}) \cap LE_i(\underline{\pi}) = \emptyset$ and $i \in AV_j(\underline{\pi})$ so that $i \in W_{ij}(\underline{\pi})$.

Based on these index sets, the following index functions are now introduced.

$$I_{ij}(\underline{\pi}) = \begin{cases} 1 & \text{if} \quad |W_{ij}(\underline{\pi})| = 1 \\ 0 & \text{else} \end{cases} \tag{4.2.6}$$

$$J_{ij}(\underline{\pi}) = \begin{cases} 1 & \text{if} \quad |W_{ij}(\underline{\pi})| > 1 \\ 0 & \text{else} \end{cases} \tag{4.2.7}$$

where $|A|$ denotes the cardinality of a set A. It should be noted from (4.2.5) that if $W_{ij}(\underline{\pi}) \neq \emptyset$ then $i \in AV_j(\underline{\pi})$ so that $i \in W_{ij}(\underline{\pi})$. Hence if $I_{ij}(\underline{\pi}) = 1$, then $W_{ij}(\underline{\pi}) = \{i\}$, i.e. $I_{ij}(\underline{\pi}) = 1$ if and only if supplier i exclusively provides the service to customer j. Similarly, one has $J_{ij}(\underline{\pi}) = 1$ if and only if supplier i jointly provides the service to customer j with other suppliers. When a price vector $\underline{\pi} \stackrel{\text{def}}{=} [\pi_1, \pi_2, \cdots, \pi_M]^T \in S$ is given, the payoff function of supplier i is then given by

$$P_i(\underline{\pi}) = \sum_{j \in \mathcal{N}} D_j(\pi_i - c_{ij}) \left\{ I_{ij}(\underline{\pi}) + \frac{J_{ij}(\underline{\pi})}{|W_{ij}(\underline{\pi})|} \right\} \quad \text{for all} \quad i \in \mathcal{M} \tag{4.2.8}$$

where $J_{ij}(\underline{\pi})/|W_{ij}(\underline{\pi})| \stackrel{\text{def}}{=} 0$ if $J_{ij}(\underline{\pi}) = 0$ and $W_{ij}(\underline{\pi}) = 0$.

The following conventional notion in game theory is employed. Given $\underline{\pi} = [\pi_1, \cdots, \pi_M]^T$, we write $\underline{\pi}_{\setminus i} = [\pi_1, \cdots, \pi_{i-1}, \pi_{i+1}, \cdots, \pi_M]^T$ and $(a_i, \underline{\pi}_{\setminus i}) = [\pi_1, \cdots, \pi_{i-1}, a_i, \pi_{i+1}, \cdots, \pi_M]^T$.

Definition 4.2.1

a) For $i \in \mathcal{M}, \pi_i^*$ is a best reply against $\underline{\pi}_{\setminus i}$ if $P_i(\pi_i^*, \underline{\pi}_{\setminus i}) = \max_{\pi_i \in S_i}[P_i(\pi_i, \underline{\pi}_{\setminus i})]$.

b) For $i \in \mathcal{M}, B_i(\underline{\pi}_{\setminus i}) = \{\pi_i^* \mid \pi_i^*$ is a best reply against $\underline{\pi}_{\setminus i}\}$ is called the set of best replies of supplier i against $\underline{\pi}_{\setminus i}$.

c) The best reply correspondence $B : S \to S$ is defined as $B(\underline{\pi}) = \prod_{i=1}^{M} B_i(\underline{\pi}_{\setminus i})$.

d) $\underline{\pi}^*$ is a Nash equilibrium, denoted by $\underline{\pi}^* \in \mathcal{NE}$, if and only if $\underline{\pi}^* \in B(\underline{\pi}^*)$.

Of interest is to see whether one or more than one Nash equilibrium points exist, i.e. $\mathcal{NE} \neq \emptyset$. In the next section, a necessary and sufficient condition is given under which $\mathcal{NE} \neq \emptyset$. An example is provided for illustrating this case. This condition is rather restrictive however and normally one has $\mathcal{NE} = \emptyset$. Section 4 is devoted to exhibit typical strategies of suppliers, when $\mathcal{NE} = \emptyset$, through a numerical example.

4.3 A Necessary and Sufficient Condition for Existence of Nash Equilibrium

In this section we prove a necessary and sufficient condition under which Nash Equilibriums exist for the model defined in the previous section. A few preliminary lemmas are needed. The first lemma states that, if supplier i is the only supplier to service customer j when all suppliers offer the maximum price U, then supplier i remains to be the unique supplier to customer j for any price vector as long as supplier i could generate a positive return from π_i.

Lemma 4.3.1 *Let $\underline{U} = [U, \cdots, U]$. If $AV_j(\underline{U}) = \{i\}$ for some $j \in \mathcal{N}$, then, for any price vector $\underline{\pi}$ satisfying $i \in AV_j(\underline{\pi})$, one has $W_{ij}(\underline{\pi}) = \{i\}$.*

The next lemma states that if $\underline{\pi} \neq \underline{U}$, then at least one supplier could serve at least one customer with price less than the upper limit U.

Lemma 4.3.2 *If $\underline{\pi}$ satisfies the condition in (4.2.1) and $\underline{\pi} \neq \underline{U}$, then there exists at least one pair of supplier i and customer j such that $|W_{ij}(\underline{\pi})| \geq 1$ and $\pi_i < U$.*

The third and last lemma implies that if supplier i is the unique supplier for customer j, then supplier i could increase its price, while remaining to be the single service provider to customer j, as long as the increased price is less than the nearest price of the competitors.

Lemma 4.3.3 *For $\underline{\pi}^* = [\pi_1^*, \pi_2^*, \cdots, \pi_M^*]$ with $\pi_i^* < U$ for some $i \in \mathcal{M}$, let $\Delta > 0$ be sufficiently small so that*

$$\pi_i^\sharp \overset{def}{=} \pi_i^* + \Delta < \min_{m \in LA_i(\underline{\pi}^*)} \{\pi_m^*\} \quad . \tag{4.3.1}$$

Then the following statements hold true for all $j \in \mathcal{N}$.

1) $|W_{ij}(\pi_i^{\sharp}, \underline{\pi}_{\setminus i}^{*})| \leq 1$

2) *If* $|W_{ij}(\underline{\pi}^{*})| = 1$, *then* $|W_{ij}(\pi_i^{\sharp}, \underline{\pi}_{\setminus i}^{*})| = 1$

We are now in a position to prove the main theorem of this section.

Theorem 4.3.4 *For the game defined in Section 4.2, the following two statements hold true.*

1) $\mathcal{NE} \neq \emptyset$ *if and only if* $|AV_j(\underline{U})| \leq 1$ *for all* $j \in \mathcal{N}$

2) *If* $\mathcal{NE} \neq \emptyset$, *then* $\mathcal{NE} = \{\underline{U}\}$

From Theorem 4.3.4, one sees that $\underline{U}$ is the only candidate to be the Nash equilibrium. If $\underline{U}$ is not Nash equilibrium, then this game has no equilibriums. In this case, the market is completely separated by the suppliers, where there is only one supplier for each customer. The rest of the suppliers cannot offer the customer since the cost is above the upperbound price. However, this situation is rather unnatural. In the next section we show the case of $\mathcal{NE} = \emptyset$ and illustrate how players may continue to behave forever in a cyclic manner in pursuit of maximizing their profits.

4.4 Cyclic Phenomenon in Case of Non-Existence of Nash Equilibrium

In this section, we illustrate typical strategies of suppliers, when $\mathcal{NE} = \emptyset$. We assume that there are two suppliers and three customers, where $U = 50\,(\text{Yen/m}^3)$, $D_1 = 100\,(\text{Mcm/y})$, $D_2 = 200\,(\text{Mcm/y})$ and $D_3 = 150\,(\text{Mcm/y})$ (Mcm/y=thousand cubic meter per year). The transportation costs c_{ij} are given in Table 4.4.1. Theorem 4.3.4 shows that, if $\underline{U} \notin \mathcal{NE}$, this game has no Nash equilibriums. In this example, each supplier tries to obtain the furthest customer demand by setting lower price than its competitor. This supplier acquires the

Table 4.4.1: The Values of c_{ij} When $\mathcal{NE} = \emptyset$

i \ j	1	2	3
1	37	40	44
2	35	47	40

Table 4.4.2: Each Supplier's Behaviour When $\mathcal{NE} = \emptyset$

	Offering Price		Acquired Customers		Payoff Value (Thousand YEN)	
Step \ Supplier	1	2	1	2	1	2
Initial	50	50	1,2,3	1,2,3	2,100	1,800
1st	49	50	1,2,3	None	3,750	0
2nd	49	48	None	1,2,3	0	2,700
3rd	47	48	1,2,3	None	2,850	0
4th	47	46	2	1,3	1,400	2,000
5th	50	46	2	1,3	2,000	2,000
6th	50	49	None	1,2,3	0	3,150
7th	48	49	1,2,3	None	3,300	0
8th	48	47	2	1,3	1,600	2,250
9th	46	47	1,2,3	None	2,400	0
10th	46	45	2	1,3	1,200	1,750
11th	50	45	2	1,3	2,000	1,750
12th	50	49	None	1,2,3	0	3,150

new distant customer at the expense of losing profits of the existing near customers since each supplier must set the same delivery price to all customers. We show this situation through a numerical example. Since $|AV_j(\underline{U})| = 2 > 1$ for all $j = 1, 2, 3$, one has $\mathcal{NE} = \emptyset$ from Theorem 4.3.4. Let $\underline{\pi} = \underline{U}$ be an initial price vector. For the sake of convenience, we discretize the strategy set so that each supplier can only take integer prices, and suppose each supplier changes its strategy in turn so as to maximize its profit. Table 4.4.2 and Figure 4.4.1 show the cyclic behavior of each supplier under the condition of Table 4.4.1. Here the initial price vector is $\underline{U} = [50\,,\, 50]$, and the first action is taken by Supplier 1. At the first step, Supplier 1 tries to maximize its profit by setting the price of 49 (Yen/m^3) lower than

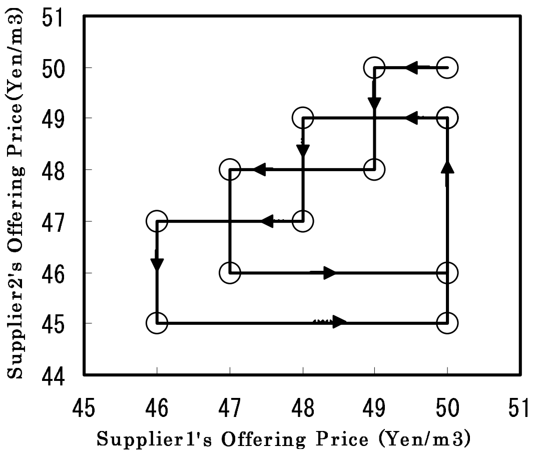

Figure 4.4.1: Cyclic Phenomenon with 2 Supplier and 3 Customer Model when $\mathcal{NE} = \emptyset$

its competitor and to eliminate Supplier 2. In return, Supplier 2 also takes a similar action by setting the price of 48 (Yen/m^3). This process continues several times. At the 4th step, Supplier 2 has no choice but to set the lower price of 46 (Yen/m^3) to secure Customers 1 and 3 at the expense of giving up Customer 2. Since it does not result in a positive return to provide service to Customer 2 at the price of 46 (Yen/m^3), Supplier 2 cannot offer the service to Customer 2. However it is better off to acquire the other customers even with low average earning per unit instead of losing all customers or splitting demands of all customers. At this point, Supplier 1 already monopolizes Customer 2, and it is in a position to enjoy the highest per-unit earning without losing the customer by setting the upper-bound price of 50 (Yen/m^3). And this cyclic process is repeated indefinitely.

Chapter 5

Structural Analysis of Two Person Game with Mixed Strategy for Delivery Services

5.1 Introduction

In this chapter, the competitive market model for a homogeneous service discussed in Chapter 4 is again addressed, where the optimal pricing strategy among mixed strategies is analyzed. In order to assure analytical tractability, we limit ourselves to two suppliers and two customers with complete symmetry. As shown in Chapter 4, this game has the unique Nash equilibrium of pure strategy type under the condition that each supplier secures the nearest customer in monopoly. If the delivery service areas of the two suppliers overlap each other, there exists no equilibrium within pure strategies. To the best knowledge of the author, the literature discussed in Chapter 4 largely focuses on pure strategies and analysis for mixed strategies has been ignored. However, the role of mixed strategies has been increasing its importance in analyzing the energy supply industry in Japan. This is so because the price table is not revised frequently even after the deregulation in Japan. Accordingly, it is necessary to read pricing strategies of competitors at the time of bidding, which demands to device a mixed strategy since reading the competitors' strategies involves uncertainty.

The purpose of this chapter is to fill this gap by developing a duopoly model with two symmetric customers and to construct the Nash equilibriums explicitly when mixed strategies are defined on a finite set of L discrete points that are chosen in such a way that their reciprocals are equally distanced in a finite interval. The limiting strategies as $L \to \infty$ are also derived explicitly. It is shown that these limiting strategies are Nash equilibriums within the context of mixed strategies defined on continuum.

The structure of this chapter is as follows. In Secion 5.2, a duopoly model with two symmetric customers is introduced and a game-theoretic framework is described formally. By choosing discrete pricing points in a peculiar way, the Nash equilibriums are constructed explicitly in Section 5.3. Section 5.4 is devoted to analysis of the limiting behavior of the strategies derived in Section 5.3 as $L \to \infty$ and prove that the limit of the Nash equilibrium in the discrete model is also the Nash equilibrium in the original model. In the last section, numerical examples are presented and managerial implications are discussed.

5.2 Model Description

We consider a market consisting of two suppliers and two customers, where each supplier provides a homogeneous service such as natural gas transported by LNG tank lorry for industrial use. Each customer may represent one large industry or a group of residents in the same district. For convenience, the near customer of supplier i is defined as customer i and the distant customer as customer $3 - i$, $i = 1, 2$ as depicted in Figure 5.2.1. The market is assumed to be symmetric in that a) both suppliers have the same costs c_{high} and c_{low} for providing service to the distant customer and the near customer respectively, where $c_{high} < c_{low}$; b) both customers have the same demand D; and c) each supplier has to offer a uniform price upon delivery to both of the two customers despite the cost difference. Each supplier provides its service only when it results in a positive return to do so and each

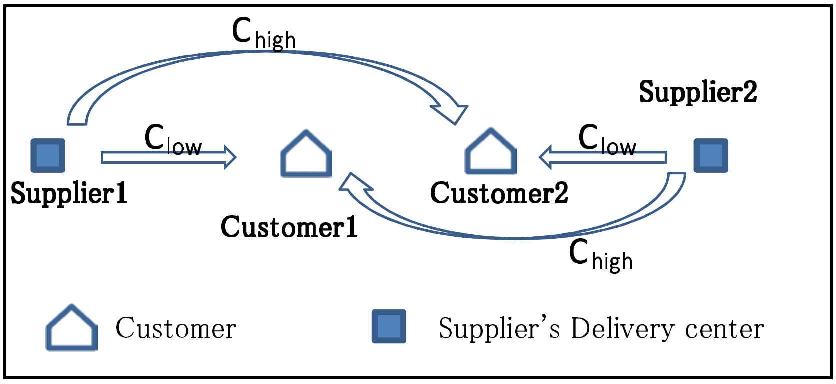

Figure 5.2.1: Two Supplier Two Customer Model

customer chooses the supplier which offers the lower price. When the two suppliers happen to offer the same price to a customer, the demand of the customer is split evenly between the two suppliers. Since the service under consideration is typically an energy supply service, it is also natural to assume that there exists a price upper bound U. It should be noted that, if $c_{low} < \pi_i \leq c_{high}$, supplier i monopolizes its near customer and the price can be increased to c_{high} without losing its monopoly of the near customer. Accordingly, one has $\pi_i \in I = [c_{high}, U]$ for $i = 1, 2$ where π_i is the uniform price offered by supplier i. In what follows, we describe a game structure defined on the strategy set I.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let RV be a set of random variables defined on $(\Omega, \mathcal{F}, P)$ with full support on $I = [c_{high}, U]$. A mixed strategy of supplier i then corresponds to a random variable $X_i \in RV$. Each supplier decides its strategy independently of the other and each supplier has enough production capacity to meet customers' demands. Given $\pi_1 = X_1(\omega_1)$ and $\pi_2 = X_2(\omega_2)$ for some $\omega_1, \omega_2 \in \Omega$, the payoff function of supplier i is given by

$$h_i(\pi_1, \pi_2) = \begin{cases} 2(\pi_i - c_{mid})D, & \pi_i < \pi_{3-i}, \quad \pi_i, \pi_{3-i} \in (c_{high}, U] \\ (\pi_i - c_{mid})D, & \pi_i = \pi_{3-i}, \quad \pi_i, \pi_{3-i} \in (c_{high}, U] \\ 0, & \pi_i > \pi_{3-i}, \quad \pi_i, \pi_{3-i} \in (c_{high}, U] \\ (c_{high} - c_{low})D, & \pi_i = c_{high}, \quad \pi_{3-i} \in (c_{high}, U] \\ (\pi_i - c_{low})D, & \pi_i \in (c_{high}, U], \quad \pi_{3-i} = c_{high} \end{cases}, \tag{5.2.1}$$

where $c_{mid} \stackrel{\text{def}}{=} (c_{low} + c_{high})/2$. If $c_{high} < \pi_i < \pi_{3-i} \leq U$, supplier i can monopolize the entire market with demand $2D$ at the average earning per unit of $\pi_i - c_{mid}$. When $c_{high} < \pi_i = \pi_{3-i} \leq U$, the demand D of each customer is split evenly between the two suppliers and the average earning per unit is again $\pi_i - c_{mid}$. The case that $c_{high} < \pi_{3-i} < \pi_i \leq U$ is the opposite of the first case and the competitor of supplier i monopolizes the entire market. For the case of $c_{high} = \pi_i < \pi_{3-i} \leq U$, supplier i can not produce a positive profit from the distant customer and therefore captures only the near customer with average earning per unit of $c_{high} - c_{low}$. Finally, if $c_{high} = \pi_{3-i} < \pi_i \leq u$, supplier i is forced to settle for the near customer with the average earning per unit of $\pi_i - c_{low}$.

Let S_i be the strategy set of supplier i and define $S = S_1 \times S_2$. In our model, one has $S_1 = S_2 = RV$ so that $S = RV \times RV$. Given $(X_1, X_2) \in S$, let $V_i(X_1, X_2) = E[h_i(X_1, X_2)]$ be the expected payoff function of supplier i.

In the next section, we focus on discrete random variables in RV when the discrete support points are chosen in such a way that their reciprocals are separated by equal distance, and two types of Nash equilibriums for the descretized game are constructed explicitly. It is shown in Section 5.4 that the limiting mixed strategies are also the Nash equilibriums for the original game defined on continuum when the equal distance diminishes to 0.

5.3 Nash Equilibriums with Specific Discrete Support

In this section, we provide a constructive proof for the existence of Nash equilibriums by discretizing the game defined in Secion 5.2.

Let $\underline{a} = [a_1, \cdots, a_L]^T \in \mathcal{R}^L$ $(L \geq 2)$ be such that

$$a_1 = (c_{high} - c_{mid})D\,; \tag{5.3.1}$$

$$\frac{1}{a_m} = (L-m)\Delta + \frac{1}{a_L}, m \in \mathcal{L} \setminus \{1\}; \text{and} \tag{5.3.2}$$

$$a_L = (U - c_{mid})D\ , \tag{5.3.3}$$

$$\text{where} \quad K = \frac{1}{a_1} - \frac{1}{a_L}, \quad \Delta = \frac{K}{L - \frac{3}{2}}\ , \quad \mathcal{L} = \{1, 2, 3, \cdots, L\} \quad . \tag{5.3.4}$$

It should be noted that $\underline{a}$ is constructed in such a way that

$$\frac{1}{a_m} - \frac{1}{a_{m+1}} = \Delta, m \in \mathcal{L} \setminus \{1, L\}; \tag{5.3.5}$$

$$\frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{2}\Delta \quad . \tag{5.3.6}$$

We now define $\underline{v}_L = [v_1, \cdots, v_L] \in \mathcal{R}^L$ in terms of $\underline{a}$ as

$$\underline{v}_L = \frac{1}{D}\underline{a} + c_{mid}\underline{1}_L \qquad , \tag{5.3.7}$$

where $\underline{1}_m$ is the m-dimensional vector whose components are all 1. We note that $v_1 = c_{high} < v_2 < v_3 < \cdots v_{L-1} < v_L = U$.

Let $DRV(\underline{v}_L)$ be a set of discrete random variables with full support on $\{v_1, \cdots, v_L\}$, where $X \in DRV(\underline{v}_L)$ is represented by a probability vector $\underline{q}$ with $P[X = v_m] = q_m$, $m \in \mathcal{L}$, and we write $X \in DRV(\underline{v}_L)$ or $\underline{q} \in DRV(\underline{v}_L)$ interchangeably.

The decomposition of the interval $[c_{high}, U]$ by $\underline{v}_L$ is depicted in Figure 5.3.1. Let $\underline{\underline{H}}_i = [h_i(v_m, v_n)]_{m,n \in \mathcal{L}}, i = 1, 2$ with $h_i(v_m, v_n)$ as given in (5.2.1). One sees that $V_i(\underline{q}_1, \underline{q}_2) = \underline{q}_1^T \underline{\underline{H}}_i\, \underline{q}_2$, $i = 1, 2$. From (5.2.1), it can be seen that $h_1(\pi_1, \pi_2) = h_2(\pi_2, \pi_1)$ so that $\underline{\underline{H}}_2 = \underline{\underline{H}}_1^T$.

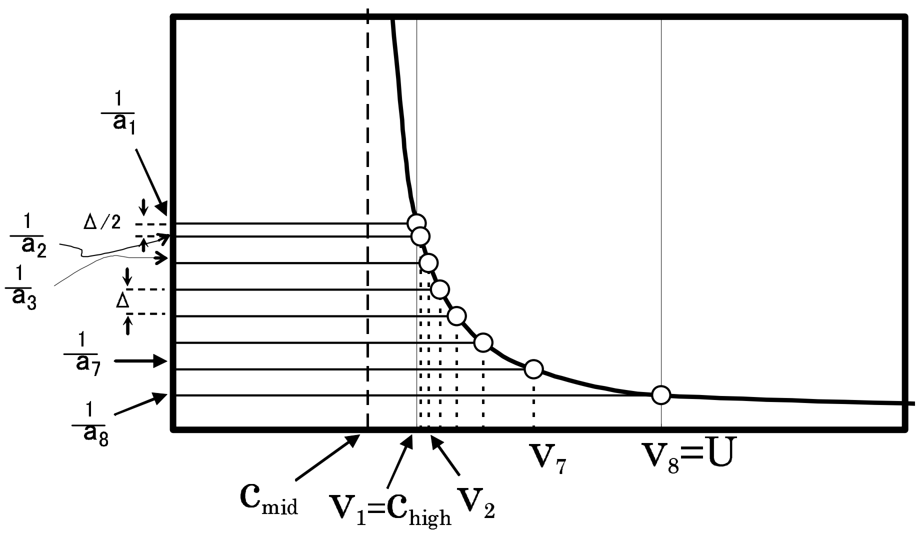

Figure 5.3.1: The Decomposition of the Interval

It then follows that $V_2(\underline{q}_1, \underline{q}_2) = \underline{q}_1^T \underline{\underline{H}}_2\, \underline{q}_2 = \underline{q}_2^T \underline{\underline{H}}_2^T \underline{q}_1 = \underline{q}_2^T \underline{\underline{H}}_1\, \underline{q}_1$. Hence, it is possible to define $V_i(\underline{q}_1, \underline{q}_2)$ as

$$V_i(\underline{q}_1, \underline{q}_2) = \underline{q}_i^T \underline{\underline{H}}\, \underline{q}_{3-i} \text{ for } i = 1, 2 \tag{5.3.8}$$

$$\text{where } \underline{\underline{H}} \stackrel{def}{=} [h_1(v_m, v_n)]_{m,n\in\mathcal{L}} = \underline{\underline{H}}_1 . \tag{5.3.9}$$

We then introduce the following notation.

Definition 5.3.1

a) $\alpha_1 = \frac{2a_1}{C_1}(\frac{2}{a_L} - \Delta)$, $\alpha_2 = \frac{2a_1}{C_1}\Delta$, $\alpha_3 = \frac{\frac{2}{a_L}}{\frac{1}{a_1}+\frac{1}{a_L}}$, $\alpha_4 = \frac{2\Delta}{\frac{1}{a_1}+\frac{1}{a_L}}$, $\alpha_5 = a_1\Delta$, $\alpha_6 = a_1(\frac{1}{a_L} + \frac{\Delta}{2})$
$C_1 = 2(\frac{a_1}{a_L} + 1) - a_1\Delta$

b) $\underline{f} \in \mathcal{R}^{L-1}$ *as* $(\underline{f})_m = \{1 + (-1)^m\}/2, m \in \mathcal{L} \setminus \{L\}$

c) $\hat{\underline{e}}_m \in \mathcal{R}^{L-1}$ and $\underline{e}_m \in \mathcal{R}^L$ are $m-$th unit vectors in $\mathcal{R}^{L-1}$ and $\mathcal{R}^L$ respectively.

d) $\underline{w}(x, y) = x\underline{1}_{L-1} + (y - x)\hat{\underline{e}}_{L-1} \in \mathcal{R}^{L-1}$

e) $\mathcal{NE}(\underline{v}_L)$ *is a set of Nash equilibriums of the discretizing game defined in this section.*

We show the first type of Nash equilibrium of this section in the following theorem, and proof is given in Appendix B.

Theorem 5.3.2 *Define* $\underline{q}^{*T} = [\alpha_1, \alpha_2 \underline{1}_{L-1}^T]$. *If* $L > \frac{a_L}{2a_1} + 1$, *then* $(\underline{q}^*, \underline{q}^*) \in \mathcal{NE}(\underline{v}_L)$. *The payoff values are* $V_1(\underline{q}^*, \underline{q}^*) = V_2(\underline{q}^*, \underline{q}^*) = D(c_{high} - c_{low})$.

This theorem states that a Nash equilibrium can be achieved when the two suppliers offer the same mixed strategy $\underline{q}^{*T} = [\alpha_1, \alpha_2 \underline{1}_{L-1}^T] \in DRV(\underline{v}_L)$. One sees from Definition 5.3.1 that α_1 is much larger than α_2 for large values of L. This means that it is necessary to assign a higher probability of α_1 to $v_1 = c_{high}$ to secure near customer, and at the same time it is also crucial to allocate a small but positive probability α_2 to all other price alternatives so that $(\underline{q}^*, \underline{q}^*) \in \mathcal{NE}(\underline{v}_L)$ can be assured. Next theorem shows that there exists a different type of Nash equilibrium $(\underline{q}^\sharp, \underline{q}^\dagger) \in \mathcal{NE}(\underline{v}_L)$, where the two suppliers take different mixed strategies, and one of the two player's payoff is the same as that of Theorem 5.3.2. As before proof is given in Appendix.

Theorem 5.3.3 *Define* $\underline{q}^{\sharp T} \stackrel{def}{=} \frac{4}{4-\alpha_4}[\alpha_3, \alpha_4 \underline{f}^T]$ *and* $\underline{q}^{\dagger T} \stackrel{def}{=} [0, \underline{w}^T(\alpha_5, \alpha_6)]$. *If* L *is even and* $L \geq 4$, *then* $(\underline{q}^\sharp, \underline{q}^\dagger), (\underline{q}^\dagger, \underline{q}^\sharp) \in \mathcal{NE}(\underline{v}_L)$. *The payoff values at this equilibrium are given as* $V_1(\underline{q}^\sharp, \underline{q}^\dagger)(= V_2(\underline{q}^\dagger, \underline{q}^\sharp)) = D(c_{high} - c_{low})$, $V_2(\underline{q}^\sharp, \underline{q}^\dagger)(= V_1(\underline{q}^\dagger, \underline{q}^\sharp)) = \frac{4}{4-\alpha 4} D(c_{high} - c_{low})$.

It should be noted that, as we will see, one has $\lim_{L\to\infty} \underline{q}^* = \lim_{L\to\infty} \underline{q}^\sharp$, while $\lim_{L\to\infty} \underline{q}^\dagger$ is quite different. The supplier with $\underline{q}^\sharp$ tends to protect the near customer by offering lower prices with higher probabilities, while the strategy with $\underline{q}^\dagger$ places importance of acquiring both customers by expanding the service area.

5.4 Limit Theorems of Nash Equilibriums with Specific Discrete Support

In the previous section, three Nash equilibriums $(\underline{q}^*, \underline{q}^*)$, $(\underline{q}^\sharp, \underline{q}^\dagger)$ and $(\underline{q}^\dagger, \underline{q}^\sharp)$ are constructed explicitly, when the strategy set consists of L discrete supporting points for pricing with $\underline{v}_L = [v_{L:1}, \cdots, v_{L:L}]$ as given in (5.3.7), where we write each compenent of $\underline{v}_L$ as $[v_{L:1}, \cdots, v_{L:L}]$

instead of $[v_1, \cdots, v_L]$ throughout this section to emphasize the demesion of $\underline{v}_L$. The purpose of this section is to analyze the limiting behaviour when $L \to \infty$, and focus on the situation where L is large enough. Therefore in the remaider of this section we assume that L is even and $L > \max\{2, \frac{a_L}{2a_1} + 1\}$. We write $\tilde{L} \to \infty$ ($\tilde{L} \stackrel{\text{def}}{=} L/2$) instead of $L \to \infty$ to clarify that L moves toward infinity in a set of even numbers.

Let X_L^*,$X_L^\sharp$ and $X_L^\dagger$ be discrete random variables associated with $\underline{q}^*$,$\underline{q}^\sharp$ and $\underline{q}^\dagger$ respectively, which are given by Defitition 5.4.1 c). Next two theorems (proof is provided in Appendix) show that these random variables converge to some mixed strategies (X^*, X^*), $(X^*, X^\dagger)$ and $(X^\dagger, X^*)$ in $S = RV \times RV$ of the original problem as $\tilde{L} \to \infty$, and they are also Nash equilibriums. We then introduce the following notation.

Definition 5.4.1

a) $\alpha_{1:\infty} = \lim_{\tilde{L}\to\infty} \alpha_1 (= 2a_1/(a_1 + a_L)) \qquad \alpha_{6:\infty} = \lim_{\tilde{L}\to\infty} \alpha_6 (= a_1/a_L)$

b) $F^*_\infty(x)$ *and* $F^\dagger_\infty(x)$ *are distribution functions defined on* $[c_{high}, U]$ *given by*

$$F^*_\infty(x) = \alpha_{1:\infty} + \frac{(1-\alpha_{1:\infty})}{KD}\left(\frac{1}{c_{high}-c_{mid}} - \frac{1}{x-c_{mid}}\right)$$

$$F^\dagger_\infty(x) = \begin{cases} \frac{(1-\alpha_{6:\infty})}{KD}\left(\frac{1}{c_{high}-c_{mid}} - \frac{1}{x-c_{mid}}\right), & c_{high} \le x < U \\ 1 \qquad\qquad\qquad\qquad\qquad\qquad\quad , & x = U \end{cases}$$

c) $r^*_{L:m} = \sum_{m'=1}^{m} q^*_{L:m'}$; $r^\sharp_{L:m} = \sum_{m'=1}^{m} q^\sharp_{L:m'}$; *and* $r^\dagger_{L:m} = \sum_{m'=1}^{m} q^\dagger_{L:m'}$ *where* $\underline{q}^* = [q^*_{L:1}, \cdots, q^*_{L:L}]^T$; $\underline{q}^\dagger = [q^\dagger_{L:1}, \cdots, q^\dagger_{L:L}]^T$; *and* $\underline{q}^\sharp = [q^\sharp_{L:1}, \cdots, q^\sharp_{L:L}]^T$ *are as in Theorem 5.3.2 and 5.3.3.*

c) $(\Omega, \mathcal{F}, P)$ *is a probability space where* $\Omega = (0, 1]$, $\mathcal{F}$ *is a Borel field on* $\Omega = (0, 1]$ *and* P *is the one-dimensional uniform probability measure on* $(\Omega, \mathcal{F})$.

e) $\{\Omega^*_{L:m}\}$, $\{\Omega^\sharp_{L:m}\}$ *and* $\{\Omega^\dagger_{L:m}\}$ *for* $m \in \mathcal{L}$ *are partitions of* Ω *given by* $\Omega^*_{L:1} = (0\ ,\ r^*_{L:1}]$; *and* $\Omega^*_{L:m} = \left(r^*_{L:m-1}\ ,\ r^*_{L:m}\right]$ *for* $m = 2, \cdots, L$. $\{\Omega^\sharp_{L:m}\}$ *and* $\{\Omega^\dagger_{L:m}\}$ *are defined similarly.*

f) $X^*_L(\omega), X^\sharp_L(\omega), X^\dagger_L(\omega) \in DRV(\underline{v}_L)$ *are the random variables defined on* $(\Omega, \mathcal{F}, P)$ *given by* $X^*_L(\omega) = v_{L:m}$ *if* $\omega \in \Omega^*_{L:m}, m = 1, 2, \cdots, L$. $X^\sharp_L(\omega)$ *and* $X^\dagger_L(\omega)$ *are defined in a similar way* *by replacing* $\Omega^*_{L:m}$ *by* $\Omega^\sharp_{L:m}$ *and* $\Omega^\dagger_{L:m}$.

g) $X^*(\omega), X^\dagger(\omega) \in RV$ *are the random variables defined on* $(\Omega, \mathcal{F}, P)$ *given by*

$$\begin{aligned} X^*(\omega) &= \begin{cases} v_{L:1} & \text{if } \omega \in (0, \alpha_{1:\infty}] \\ F^{*-1}_\infty(\omega) & \text{if } \omega \in (\alpha_{1:\infty}, 1] \end{cases} \\ X^\dagger(\omega) &= \begin{cases} F^{\dagger-1}_\infty(\omega) & \text{if } \omega \in (0\,,\ 1-\alpha_{6:\infty}] \\ v_{L:L} & \text{if } \omega \in (1-\alpha_{6:\infty}\,,\ 1] \end{cases}. \end{aligned}$$

h) $\mathcal{NE}$ *is a set of Nash equilibriums of the game defined in Section 5.2*

Theorem 5.4.2

a) $F^*_\infty(x)$ *and* $F^\dagger_\infty(x)$ *are the distribution functions of* $X^*(\omega)$ *and* $X^\dagger(\omega)$ *repectively.*

b) $\underline{q}^*$, $\underline{q}^\sharp$ *and* $\underline{q}^\dagger$ *are the probability vectors of* $X^*_L(\omega)$, $X^\sharp_L(\omega)$ *and* $X^\dagger_L(\omega)$ *repectively.*

c) $X^*_L(\omega) \overset{a.e.}{\to} X^*(\omega)\,, X^\sharp_L(\omega) \overset{a.e.}{\to} X^*(\omega)$; *and* $X^\dagger_L(\omega) \overset{a.e.}{\to} X^\dagger(\omega)$ *as* $\tilde{L} \to \infty$

where " $\overset{a.e.}{\to}$ " *denotes the almost everywhere convergence.*

In what follows, we prove that these limiting strategies are also Nash equilibriums for the original game. For this purpose, we need to deal with the limiting behavior of $V_1(X^*_{1,L}, X^*_{2,L})$. In the remainder of this section, we write $X^*_{i,L}, X^\sharp_{i,L}, X^\dagger_{i,L}$; and $X^*_i, X^\dagger_i, i = 1,2$ instead of $X^*_L, X^\sharp_L, X^\dagger_L$; and $X^*, X^\dagger$ to emphasize the player of the stategies. Since $h_i(\pi_1, \pi_2)$ in (5.2.1) is not continuous function of π_1, π_2, it does not, in general, hold that $\lim_{\tilde{L}\to\infty} E[h_i(X_L, Y_L)] = E[h_i(X, Y)]$ even if X_L and Y_L converge almost everywhere to X and Y as $\tilde{L} \to \infty$. A preliminary lemma is needed and a proof is provided in Appendix.

Lemma 5.4.3 *For* $i = 1, 2$ *let* Y_i *be any independently and identically distributed (i.i.d.) random variables in* RV, *and define the associated pairs of i.i.d. random variables* $Y_{i,L}$ *by* $Y_{i,L}(\omega_i) = v_{L:1}$ *if* $\omega \in \{\omega_i | Y(\omega_i) = v_{L:1}\}$ *and* $Y_{i,L}(\omega_i) = v_{L:m}$ *if* $\omega \in \{\omega_i | v_{L:m-1} < Y(\omega_i) \le v_{L:m}\}$ *for* $m = 2, 3, \cdots, L$, *where we write* ω_i *instead of* ω *to emphasize the player. Then the following statements hold;*

a) $Y_{i,L} \overset{a.e.}{\to} Y_i$ *as* $\tilde{L} \to \infty$ *for* $i = 1, 2$

b) $\lim_{\tilde{L}\to\infty} V_i(X^*_{1,L}, X^*_{2,L}) = V_i(X^*_1, X^*_2)$, $i = 1, 2$

c) $\lim_{\tilde{L}\to\infty} V_1(Y_{1,L}, X^*_{2,L}) = V_1(Y_1, X^*_2)$, $\quad \lim_{\tilde{L}\to\infty} V_2(X^*_{1,L}, Y_{2,L}) = V_2(X^*_1, Y_2)$

d) $\lim_{\tilde{L}\to\infty} V_i(X^\sharp_{1,L}, X^\dagger_{2,L}) = V_i(X^*_1, X^\dagger_2)$, $i = 1, 2$

e) $\lim_{\tilde{L}\to\infty} V_1(Y_{1,L}, X^\dagger_{2,L}) = V_1(Y_1, X^\dagger_2)$, $\quad \lim_{\tilde{L}\to\infty} V_2(X^\sharp_{1,L}, Y_{2,L}) = V_2(X^*_1, Y_2)$

Theorem 5.4.4 *The following two statements hold true.*

a) $(X^*_1, X^*_2) \in \mathcal{NE}$

b) $(X^*_1, X^\dagger_2) \in \mathcal{NE}$, $(X^\dagger_1, X^*_2) \in \mathcal{NE}$

Proof: First we note that the equilibrium $(\underline{q}^*, \underline{q}^*)$ in Theorem 5.3.2 is written as $(X^*_{1,L}, X^*_{2,L})$ here. For any $Y_1 \in S_1$, define $Y_{1,L}$ as in Lemma 5.4.3. Since $(X^*_{1,L}, X^*_{2,L}) \in \mathcal{NE}(\underline{v}_L)$, one has $V_1(Y_{1,L}, X^*_{2,L}) \le V_1(X^*_{1,L}, X^*_{2,L})$ for $L \in \{2, 4, 6, \cdots\}$ so that $\lim_{\tilde{L}\to\infty} V_1(Y_{1,L}, X^*_{2,L}) \le \lim_{\tilde{L}\to\infty} V_1(X^*_{1,L}, X^*_{2,L})$. It then follows from Lemma 5.4.3 b) and c) that $V_1(Y_1, X^*_2) \le V_1(X^*_1, X^*_2)$ for all $Y_1 \in S_1$. Similarly one has $V_2(X^*_1, Y_2) \le V_2(X^*_1, X^*_2)$ for all $Y_2 \in S_2(= RV)$, proving part a).

For part b), since $(X^\sharp_{1,L}, X^\dagger_{2,L}) \in \mathcal{NE}(\underline{v}_L)$ one has $V_1(Y_{1,L}, X^\dagger_{2,L}) \le V_1(X^\sharp_{1,L}, X^\dagger_{2,L})$ for all $L > \max\{2, \dfrac{a_L}{2a_1} + 1\}$ and L is even, so that $\lim_{\tilde{L}\to\infty} V_1(Y_{1,L}, X^\dagger_{2,L}) \le \lim_{\tilde{L}\to\infty} V_1(X^\sharp_{1,L}, X^\dagger_{2,L})$. It then follows from Lemma 5.4.3 d) and e) that $V_1(Y_1, X^\dagger_2) \le V_1(X^*_1, X^\dagger_2)$ for all $Y_1 \in S_1(= RV)$. Similarly one has $V_2(X^*_1, Y_2) \le V_2(X^*_1, X^\dagger_2)$ for all $Y_2 \in S_2(= RV)$, proving that $(X^*_1, X^\dagger_2) \in \mathcal{NE}$. The fact that $(X^\dagger_1, X^*_2) \in \mathcal{NE}$ can be proven in a similar manner. □

5.5 Numerical Examples

In this section, numerical examples are provided, yielding managerial implications for energy suppliers. We consider the case that two customers are middle-sized industrial customers, receiving natural gas transported in LNG lorry tankers. It should be noted that, unlike usual city gas distribution through pipeline networks, the trasportation costs are considered to be marginal costs. Although the price and cost vary depending on the condition or demand pattern, for the sake of convenience, we suppose here $c_{low} = 40$[Yen/m^3], $c_{high} = 50$[Yen/m^3] and $U = 60$[Yen/m^3]. For energy supply within this price range, the demand price elasticity is thought to be very small.

The probabilities to win only near customer or both customers are evaluated when $(X_1^*, X_2^*) \in S$. In our model we assume each player has the same cost structure. If one player tries to secure its near customer while giving up the distance customer, it offers c_{high} since each player provides its service only when it results in a positive return. In this case, the player can capture the customer with probability one. If the player tries to capture both customers, it must offer the price $x > c_{high}$. In this case, the probabiity to win its near customer is below one since it could lose both customers when the other player offers the price between c_{high} and x. The probability can be written as $F_\infty^*(x) - F_\infty^*(c_{high})$ where F_∞^* is as in Definition 5.4.1 b). By substituting $\alpha_{1:\infty}, a_1, a_L, K$ of Definition 5.4.1a)(5.3.1)(5.3.3) and (5.3.4) into this, one has

$$F_\infty^*(x) - F_\infty^*(c_{high}) = \frac{1}{2}\frac{d_{high} + d_{low}}{d_{low}}\left(1 - \frac{d_{low} - d_{high}}{2x + d_{high} + d_{low} - 2U}\right)$$

where $d_{high} \stackrel{\text{def}}{=} U - c_{high},\ d_{low} \stackrel{\text{def}}{=} U - c_{low}$. Figure 5.5.1 and 5.5.2 depicts the winning probablity for each supplier exercising the Nash equilibtiumin (X_1^*, X_2^*) in Theorem 5.3.2. Similarly we show in Figure 5.5.3 the pobablitiy for player 1 to win both customers when the equilibrium $(X_1^*, X_2^\dagger)$ is realized. It can be seen that these winning probabilities are nonincreasing as a function of price, i.e., the higher the offering price is, the lower the two winning probabilities are. The monotonicity appears in such a way that at the price equilibrium for the mixed strategies, the expected profit is the same regardless of the offering price. However, this does not mean that the offering price is not important. It affects the winning probabilities which may be quite important for assuming the company's presence in the market.

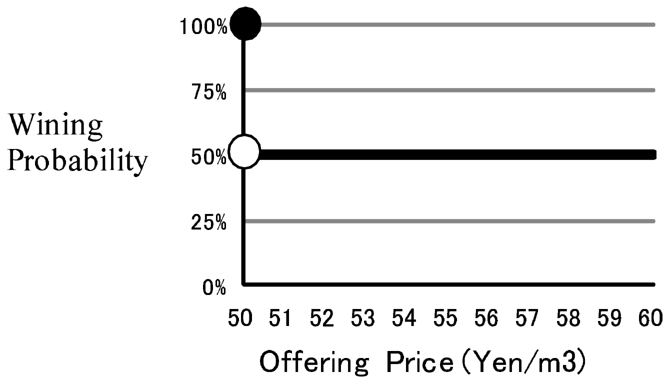

Figure 5.5.1: Probability to Win Only Near Customer When $(X_1^*, X_2^*) \in S$

It is worth noting that $X_i^*, i = 1, 2$ in Theorem 5.4.4 has the mass $m(c_{high}) = 2a_1/(a_1 + a_L)$ at c_{high}. Let $U = c_{high} + d_{high}$. From (5.3.1) and (5.3.2), one then sees that

$$m(c_{high}) = \frac{c_{high} - c_{low}}{c_{high} - c_{low} + d_{high}} \qquad . \tag{5.5.1}$$

Adopting the lowest possible price at c_{high} is the risk aversive strategy in that the supplier secures the near customer while giving up the distant customer. Equation (5.5.1) states that the mass assigned to this strategy at the limit is the ratio of the unit profit expected from the near customer under this strategy against that obtained by offering the highest possible price $U = c_{high} + d_{high}$. Clearly, the mass $m(c_{high})$ vanishes as $U \to \infty$ and the associated limiting distribution becomes absolutely continuous on $[c_{high}, \infty)$ having the probability density function given by

$$f_{\infty:U=\infty}(x) = \frac{c_{high} - c_{low}}{2} (x - c_{mid})^{-2} \qquad . \tag{5.5.2}$$

The interpretation for Theorem 5.4.4 b) can be stated as supplier i takes the risk aversive strategy by placing the mass $m_i(c_{high})$ as given in (5.5.1), while supplier $3 - i$ adopts the

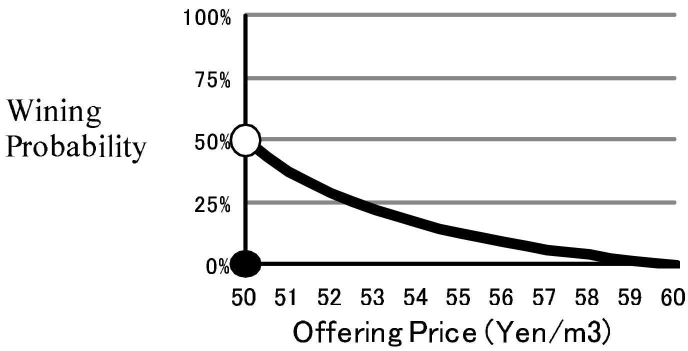

Figure 5.5.2: Probability to Win Both Customers When $(X_1^*, X_2^*) \in S$

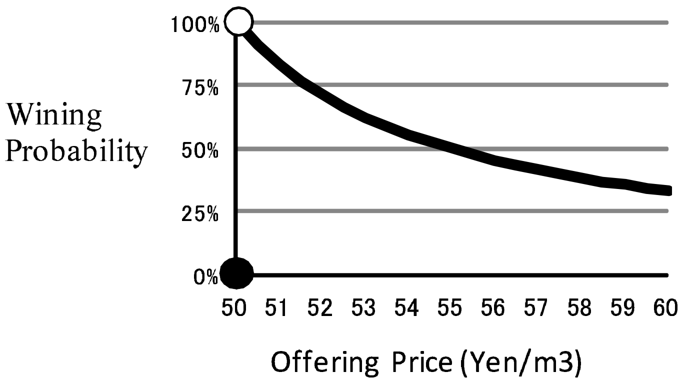

Figure 5.5.3: Probability to Win Both Customers When $(X_1^*, X_2^\dagger) \in S$

risk taking strategy by placing the mass $m_{3-i}(U)$ at the highest possible price U where

$$m_{3-i}(U) = \frac{c_{high} - c_{low}}{c_{high} - c_{low} + 2d_{high}} \quad .$$

Both $m_i(c_{high})$ and $m_{3-i}(U)$ diminish to zero as $U \to \infty$ and one observes again that both suppliers have the same associated limiting strategy specified by (5.5.2). One may then expect that there exists the unique Nash equilibrium specified by (5.5.2) with the strategy space $S = RV \times RV$ where RV is the set of all random variables defined on $[c_{high}, \infty)$. This conjecture is currently under study and will be reported elsewhere.

Chapter 6

Pipeline Investment Strategy in Response to All-Electric House Systems

6.1 Introduction

Since the beginning of the new century, all-electric house systems have been strengthening the presence in the residential energy market. In an all-electric house system, the residential energy demand within the household is supplied totally by electricity with IH (Inductive Heating) cooking heaters and an electric "heat-pump water heating and supply system." Popularity of all-electric house systems is largely due to the price-performance improvement of those equipment, as well as the strong promotion campaigns by electricity companies.

As the strategy of electricity companies for promoting all-electric house systems became clear, city gas companies have been forced to reconsider their strategy for the residential market and the small-scale commercial market. They used to enjoy the fact that residential customers tend to choose city gas for cooking and hot water if a gas pipeline is located near their houses. Because of this, if there are houses near a network of gas pipelines, the city gas company owning the network could count on the residential gas demand for a very long time. However, in the presence of all-electric house system, a residential customer may choose to adopt all-electric house system upon moving into a house even with a gas pipeline, terminating the use of city gas and hence the network of gas pipelines at that house. In response to this market change, city gas companies have been changing their investment strategy to expand their existing networks of gas pipelines, since it would require huge capital spending and a very long time for investment recovery.

The purpose of this chapter is to develop and analyze a mathematical model for capturing the cash flow of a gas company in a residential area with a given network of gas pipelines, where residents move in and move out stochastically, and choose city gas or all-electric house system upon moving in. Computational algorithms are developed for numerically evaluating the optimal number of pipelines to be installed through the cost-performance analysis of the mathematical model. Numerical examples are also given for demonstrating the efficiency of the numerical algorithms.

The structure of this chapter is as follows. In Section 6.2, a mathematical model is formally described, where residents move in and move out according to a Markov chain in continuous time, and the relevant cash flow is represented by a reward process defined on the Markov chain. In order to reflect the reality, we assume that the Markov chain and the associated cash flow are affected by an exogenous Markov chain representing the macro-economic condition. Computational algorithms are developed in Section 6.3 for numerically evaluating the expected cash flow and the expected profit at time t, enabling one to explore the optimal strategy for the number of gas pipelines to be installed at time 0. Some numerical examples are given in Section 6.4.

6.2 Model Description

We consider a newly developed residential area in which K houses can be made readily accessible to an existing network of gas pipelines at the cost of ¥cK. How these K houses may or may not be occupied would depend on the exogenous macro-economic condition. More specifically, let $J(t)$ be a birth-death process defined on $\mathcal{J} \stackrel{\text{def}}{=} \{-1, 0, 1\}$ governed by hazard rate matrix $\underline{\underline{\eta}}$ where $(\underline{\underline{\eta}})_{ij} = 0$ if $i = j$ or $i = 3, j = 1$ or $i = 1, j = 3$; and $(\underline{\underline{\eta}})_{12} = \eta^{+}_{-1}$, $(\underline{\underline{\eta}})_{23} = \eta^{+}_{0}$, $(\underline{\underline{\eta}})_{21} = \eta^{-}_{0}$, $(\underline{\underline{\eta}})_{32} = \eta^{-}_{1}$, with state -1 describing a bad economic condition, state 0 a normal economic condition, and state 1 a good economic condition. Customers arrive according to a Poisson process with parameter λ_i if $J(t) = i$. Dwell times of customers occupying one of K houses are i. i. d. (independently and identically distributed) having a common exponential distribution with mean β_i^{-1} whenever $J(t) = i$. A customer finding all K houses occupied upon his/her arrival would be lost.

In order to capture the competition of city gas against all-electric house system, it is assumed that a customer finding one of the K houses available upon his/her arrival would choose to use city gas with probability p. With probability $1 - p$, the customer decides to adopt all-electric house system. Let $M_1(t)$ be the number of houses occupying one of the K houses and using city gas at time t. Similarly, let $M_2(t)$ be the number of houses occupying one of the K houses and using all-electric house system at time t. We note that $K - \{M_1(t) + M_2(t)\}$ represents the number of vacant houses among the K houses at time t. The joint process $[M_1(t), M_2(t)]$ has the state space $\mathcal{M}_{12}$ given by

$$\begin{aligned}
\mathcal{M}_{12} &\stackrel{\text{def}}{=} \{(i,j) \mid 0 \leq i + j \leq K\} \\
&= \{(0,0), (1,0), \cdots, (K,0), (0,1), (1,1), \cdots, (K-1,1), \\
&\quad (0,2), \cdots, (1, K-1), (0,K)\} \quad . \qquad (6.2.1)
\end{aligned}$$

Excluding the installment cost ¥cK, the profit per unit time for the gas company per house using city gas is denoted by ρ.

Let $\underline{N}(t) = [J(t), M_1(t), M_2(t)]$ be the trivariate Markov chain in continuous time on

$S \stackrel{\text{def}}{=} \mathcal{J} \times \mathcal{M}_{12}$. The hazard rate matrix $\underline{\underline{\nu}}$ governing $\underline{N}(t)$ is given by

$$\underline{\underline{\nu}} = \begin{bmatrix} \underline{\underline{\xi}}_{-1} & \eta_{-1}^{+}\underline{\underline{I}} & \underline{\underline{0}} \\ \eta_0^{-}\underline{\underline{I}} & \underline{\underline{\xi}}_0 & \eta_0^{+}\underline{\underline{I}} \\ \underline{\underline{0}} & \eta_1^{-}\underline{\underline{I}} & \underline{\underline{\xi}}_1 \end{bmatrix} \tag{6.2.2}$$

where

$$\underline{\underline{\xi}}_i = \begin{bmatrix} \underline{\underline{G}}_{i:K\times K} & \underline{\underline{G}}_{i:K\times(K-1)} & \cdots & \underline{\underline{0}}_{i:K\times 1} \\ \underline{\underline{G}}_{i:(K-1)\times K} & \underline{\underline{G}}_{i:(K-1)\times(K-1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{\underline{G}}_{i:2\times 1} \\ \underline{\underline{0}}_{i:1\times K} & \cdots & \underline{\underline{G}}_{i:1\times 2} & \underline{\underline{G}}_{i:1\times 1} \end{bmatrix} \quad , \tag{6.2.3}$$

and

$$\underline{\underline{G}}_{i:m\times n} = \begin{cases} \begin{bmatrix} 0 & \lambda_i p & 0 & 0 & \cdots & 0 \\ \beta_i & 0 & \lambda_i p & 0 & \cdots & 0 \\ 0 & 2\beta_i & 0 & \lambda_i p & \cdots & 0 \\ 0 & 0 & 3\beta_i & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \lambda_i p \\ 0 & 0 & 0 & 0 & m\beta_i & 0 \end{bmatrix} & \text{if} \quad n = m \\ \begin{bmatrix} (K-m)\beta_i & 0 & 0 & \cdots & 0 \\ 0 & (K-m)\beta_i & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & (K-m)\beta_i & 0 \end{bmatrix} & \text{if} \quad n = m+1 \\ \begin{bmatrix} (1-p)\lambda_i & 0 & \cdots & 0 \\ 0 & (1-p)\lambda_i & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & (1-p)\lambda_i \\ 0 & 0 & 0 & 0 \end{bmatrix} & \text{if} \quad n = m-1 \\ \underline{\underline{0}} & \text{otherwise} \quad . \end{cases}$$

Let $Z_K(t)$ be the cumulative profit of the gas company up to time t, excluding the installment cost, given that K houses are connected to the existing network of gas pipelines at time 0. Clearly, this reward process $Z_K(t)$ can be written as

$$Z_K(t) = \rho \int_0^t M_1(t)dt \quad . \tag{6.2.4}$$

The major interest of the gas company is then to maximize the investment return over a planning horizon of time T, that is

[**Problem 6.2.1**]

Find $K^* = \arg\max\{K : E[R(K,T)]\}$,

where

$$R(K,T) = Z_K(T) - cK \quad . \tag{6.2.5}$$

Based on Equation (5.13) of Sumita and Masuda [14] combined with the uniformization procedure of Keilson [10], computational algorithms will be developed in the next section so as to solve [Problem 6.1] numerically.

6.3 Development of Computational Algorithms for Finding Optimal Number of Gas Pipelines to Be Installed

In the previous section, we have seen that the trivariate process $\underline{N}(t) = [J(t), M_1(t), M_2(t)]$ is a Markov chain in continuous time defined on $\mathcal{S} = \mathcal{J} \times \mathcal{M}_{12}$ and governed by $\underline{\underline{\nu}} = [\nu_{\underline{m}\,\underline{n}}]$, $\underline{m}\,,\underline{n} \in \mathcal{S}$ given in (6.2.2). Let $\underline{\underline{P}}(t)$ be the transition probability matrix of $\underline{N}(t)$ defined by

$$\underline{\underline{P}}(t) = [P_{\underline{m},\underline{n}}(t)]_{\ \underline{m},\underline{n}\in\mathcal{S}} \ ; \quad P_{\underline{m},\underline{n}}(t) = P[\underline{N}(t) = \underline{n}|\underline{N}(0) = \underline{m}] \quad . \tag{6.3.1}$$

From the Kolmogorov forward equation, $\underline{\underline{P}}(t)$ can be expressed in terms of the infinitesimal generator $\underline{\underline{Q}}$ of $\underline{N}(t)$. More specifically, let $\underline{\underline{\nu}}_D$ be the diagonal matrix defined by

$$\underline{\underline{\nu}}_D = [\delta_{\{\underline{m}=\underline{n}\}}\nu_{\underline{m}}]_{\ \underline{m},\underline{n}\in\mathcal{S}} \ ; \nu_{\underline{m}} = \sum_{\underline{n}\in\mathcal{S}} \nu_{\underline{m}\,\underline{n}} \tag{6.3.2}$$

and define

$$\underline{\underline{Q}} = \underline{\underline{\nu}} - \underline{\underline{\nu}}_D \tag{6.3.3}$$

One then sees that $\frac{d}{dt}\underline{\underline{P}}(t) = \underline{\underline{P}}(t)\underline{\underline{Q}}$. By solving this equation with $\underline{\underline{P}}(0) = \underline{\underline{I}}$, it then follows that

$$\underline{\underline{P}}(t) = e^{t\underline{\underline{Q}}} = \sum_{k=0}^{\infty} \frac{t^K}{k!}\underline{\underline{Q}}^k \qquad . \tag{6.3.4}$$

In principle, Equation (6.3.4) enables one to compute $\underline{\underline{P}}(t)$. Since the infinitesimal generator $\underline{\underline{Q}}$ involves negative numbers, however, the numerical procedure based on (6.3.4) may not be numerically stable, especially when the size of the space is huge. In order to overcome this difficulty, we employ the uniformization procedure of Keilson [10]. Let $\nu \geq \max_{\underline{m}\in\mathcal{S}}\{\nu_{\underline{m}}\}$ and define

$$\underline{\underline{a}}_\nu = \underline{\underline{I}} - \frac{1}{\nu}\underline{\underline{\nu}}_D + \frac{1}{\nu}\underline{\underline{\nu}} \qquad . \tag{6.3.5}$$

One sees that $\underline{\underline{a}}_\nu \geq \underline{\underline{0}}$ since the diagonal elements of $\frac{1}{\nu}\underline{\underline{\nu}}_D$ is less than or equal to 1 and $\underline{\underline{\nu}} \geq \underline{\underline{0}}$. Furthermore, $\underline{\underline{a}}_D\underline{1} = \underline{1}$ since $\underline{\underline{\nu}}_D\underline{1} = \underline{\underline{\nu}}\,\underline{1}$ from (6.3.2). Consequently, $\underline{\underline{a}}_\nu$ is a stochastic matrix. Furthermore one sees from (6.3.3) and (6.3.5) that $\underline{\underline{a}}_\nu = \underline{\underline{I}} + \frac{1}{\nu}\underline{\underline{Q}}$ or equivalently

$$\underline{\underline{Q}} = -\nu(\underline{\underline{I}} - \underline{\underline{a}}_\nu) \qquad . \tag{6.3.6}$$

Substituting (6.3.6) into (6.3.4), it then follows that

$$\underline{\underline{P}}(t) = e^{-\nu t[\underline{\underline{I}} - \underline{\underline{a}}_\nu]} = \sum_{k=0}^{\infty} e^{-\nu t}\frac{(\nu t)^k}{k!}\underline{\underline{a}}_\nu^k \qquad . \tag{6.3.7}$$

Since Equation (6.3.7) involves only non-negative numbers, the numerical procedure for computing $\underline{\underline{P}}(t)$ based on (6.3.7) is numerically stable. It may be worth noting that Equation (6.3.7) has the following probabilistic interpretation. Let $K_\nu(t)$ be the Poisson process with intensity ν. One then sees that $P[K_\nu = k] = e^{-\nu t}\frac{(\nu t)^k}{k!}$. If we define $\underline{\hat{N}}(k)$ to be the discrete time Markov chain on $\mathcal{S}$ governed by one-step transition probability matrix $\underline{\underline{a}}_\nu$, then $\underline{\underline{a}}_\nu^k$ is the k-step transition probability of $\underline{\hat{N}}(k)$. Hence, it follows that

$$\underline{N}(t) = \underline{\hat{N}}(k_\nu) \tag{6.3.8}$$

where the dwell time of $\underline{N}(t)$ in any state $\underline{m} \in \mathcal{S}$ can be "uniformized" to exponential random variate of parameter ν. In order to prepare the stochastic equivalence under iniformization, the probability of transition from $\underline{m}$ to $\underline{n}$ upon expiration of the dwell-time in state $\underline{m}$ is altered from $\frac{\nu_{\underline{m}\,\underline{n}}}{\nu_{\underline{m}}}$ to $(\underline{\underline{a}}_{\nu})_{\underline{m}\,\underline{n}}$. The expected reward $E[Z_k(t)]$ can be obtained from (5.13) of Sumita and Masuda [14]. By reinterpreting the result in our context, one sees that

$$E[Z_k(T)] = \int_0^T \underline{p}^T(t)dt \underline{\underline{\rho}}_D \underline{1} \tag{6.3.9}$$

where

$$\underline{\underline{\rho}}_D = \rho \begin{bmatrix} \underline{\underline{L}} & & \underline{\underline{0}} \\ & \underline{\underline{L}} & \\ \underline{\underline{0}} & & \underline{\underline{L}} \end{bmatrix} ; \underline{\underline{L}} = \mathrm{diag}\{0, 1, 2, \cdots, k, 0, 1, \cdots, k-1, \cdots, \cdots, 0, 1, 0\} \ . \tag{6.3.10}$$

From (6.3.7), it can be seen that

$$\int_0^T \underline{p}^T(t)dt = \sum_{k=0}^{\infty} \int_0^T e^{-\nu t} \frac{(\nu t)^k}{k!} dt \, \underline{p}^T(o) \underline{\underline{a}}_{\nu}^k \qquad . \tag{6.3.11}$$

We now define, for $k = 0, 1, \cdots$ that

$$PS(k,T) = e^{-\nu T} \frac{(\nu T)^k}{k!}; \ IN(k,T) = \int_0^T PS(k,t)dt \quad . \tag{6.3.12}$$

It can be readily seen that

$$PS(k+1,T) = \frac{\nu T}{k+1} PS(k,T) \text{ with } PS(0,T) = e^{-\nu T} \quad . \tag{6.3.13}$$

For $IN(k+1,T)$, we note that from integration by parts, that

$$\begin{aligned} IN(k+1,T) &= \frac{\nu^{k+1}}{(k+1)!} \int_0^T e^{-\nu t} t^{k+1} dt = \frac{\nu^{k+1}}{(k+1)!} \left\{ \left[-\frac{1}{\nu} e^{-\nu t} t^{k+1} \right]_0^T + \frac{k+1}{\nu} \int_0^T e^{-\nu t} t^k dt \right\} \\ &= -\frac{1}{\nu} PS(k+1,T) + IN(k,T) \quad . \end{aligned}$$

Since $IN(0,T) = \int_0^T e^{-\nu t}dt = \frac{1}{\nu}(1 - e^{-\nu T})$, the following recursion formula holds true.

$$\begin{aligned}
\underline{p}^T(t) &= \sum_{k=0}^{\infty} PS(k,t)\underline{p}^T(0)\underline{\underline{a}}_\nu^k &(6.3.14)\\
E[Z_K(T)] &= \sum_{k=0}^{\infty} IN(k,T)\underline{p}^T(0)\underline{\underline{a}}_\nu^k \underline{\underline{\rho}}_D \underline{1} &(6.3.15)\\
PS(k+1,T) &= \frac{\nu}{k+1}PS(k,T) \quad \text{with} \quad PS(0,T) = e^{-\nu T} &(6.3.16)\\
IN(k+1,T) &= -\frac{1}{\nu}PS(k+1,T) + IN(k,T) &(6.3.17)\\
&\quad \text{with} \quad IN(0,T) = -\frac{1}{\nu}(1 - e^{-\nu T})
\end{aligned}$$

Based on (6.3.14) through (6.3.17), we are now in a position to describe a computational algorithm for evaluating $E[Z_k(T)]$.

Algorithm 6.3.1

Input:

$\underline{p}^T(0)$: initial probability vector

$\underline{\underline{\nu}}$: hazard rate matrix in (6.2.2)

ν: uniformization constant in (6.3.5)

$\underline{\underline{a}}_\nu$: stochastic matrix given in (6.3.5)

$\underline{\underline{\rho}}_D$: diagonal reward matrix in (6.3.10)

T: planning horizon

Output:

0] Set $k \leftarrow 0,\ PS \leftarrow e^{-\nu T},\ IN \leftarrow -\frac{1}{\nu}(1 - e^{-\nu T}),\ ZK \leftarrow 0,\ \underline{r}^T \leftarrow \underline{p}^T(0)$

1] Loop: $ZK \leftarrow ZK + IN\underline{r}^T \underline{\underline{\rho}}_D \underline{1}$

2] $k \leftarrow k + 1$

3] $PS \leftarrow \frac{\nu}{k}PS$

4] $IN \leftarrow -\frac{1}{\nu}PS + IN$

5] If $IN\rho k < \epsilon$, stop. Otherwise set $\underline{r}^T \leftarrow \underline{r}^T\underline{\underline{a}}_\nu$ and to LOOP

We note that $E[R(K, T)]$ is increasing in T, as shown in the next proposition.

Proposition 6.3.1 $E[R(K, T)]$ *is increasing in* T*.*

Proof: From (6.3.12), it can be readily seen that $IN(k, T)$ is increasing in T for all $k \geq 0$. It then follows from (6.3.15) that $E[R(K, T)]$ is also increasing in T. $\square$

6.4 Numerical Examples

In this section, we provide numerical examples so as to demonstrate that Algorithm 6.3.1 works efficiently, enabling one to solve Problem 6.2.1 with speed. The basic values of the underlying parameters are summarized in Table 6.4.1. Throughout this section, these parameter values are assumed, unless specified otherwise. As time unit, we adopt one year. It may be worth noting that the arrival rate of customers with intention of moving into one of the K houses is one every six month when the macro-economic condition is "normal", which is increased to one every four month as the macro-economic condition improves to "good" and is decreased to one every year as it deteriorates into "bad". Once a customer moves into one of the K houses, he/she stays, on the average, ten years, which would not be affected by the macro-economic condition. The probability of choosing city gas over all-electric system is assumed to be 0.7, which is also indifferent to the changes of the macro-economic condition. If a house is occupied with city gas chosen, the investment recovery period for installing one gas pipeline is expected to be five years.

In order to estimate the birth-death process parameters for specifying the stochastic nature of the macro-economic condition, we adopt the approach by Huang and Sumita [8], where the monthly LIBOR (London Inter-Bank Offered Rate) in US dollars for the period September 1989-December 2008 was employed. By applying the maximum likelihood

Table 6.4.1: Basic Values of the Underlying Parameters

T	20
K	20
ρ	1.0
p	0.7
c	5.0

i	1	2	3
λ_i	1.0	2.0	3.0
β_i	0.1	0.1	0.1

Table 6.4.2: Birth-Death Process Parameters

η_{-1}^{+}	6.1272
η_0^{-}	9.0000
η_0^{+}	6.0000
η_1^{-}	3.1524

estimator approach for continuous time Markov chain originally proposed by Hansen [6] , the monthly estimators of Huang and Sumita[8] are converted to the yearly estimators as shown in Table 6.4.2.

Figures 6.4.1 through 6.4.3 depict $E[R(K,T)]$ as a function of K and T for $p = 0.6, 0.7$ and 0.8 respectively. We note that $E[R(K,T)]$ increases as T or p increases, as expected from Proposition 6.3.1. It seems that $E[R(K,T)]$ is concave in K. In order to observe this point more carefully, $E[R(K,T)]$ for $T = 10, 20, 30$ are plotted for $p = 0.6, 0.7$ and 0.8 in Figures 6.4.4 through 6.4.6. The concavity of $E[R(K,T)]$ in K can be seen more explicitly. The optimal values K^* for these nine cases are summarized in Table 6.4.3. One realizes that K^* increases as T or p increases.

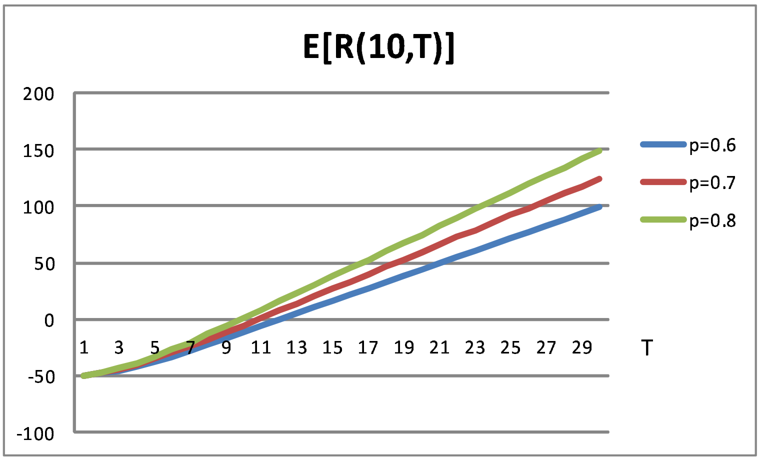

Figure 6.4.1: $\mathrm{E}[R(10, T)]$ as a Function of T

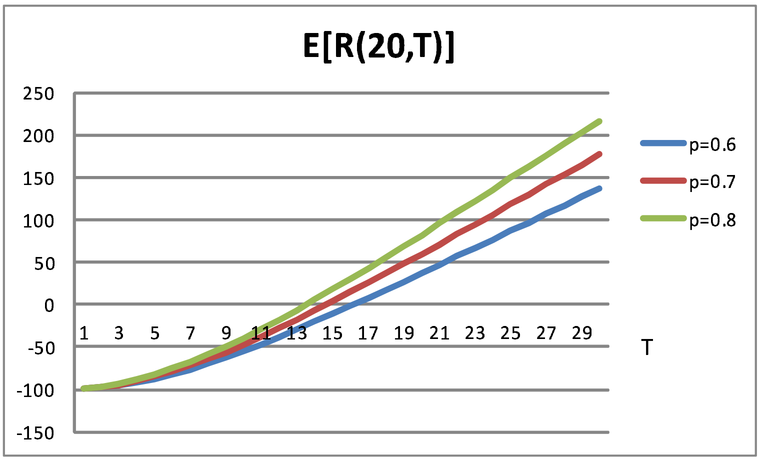

Figure 6.4.2: $\mathrm{E}[R(20, T)]$ as a Function of T

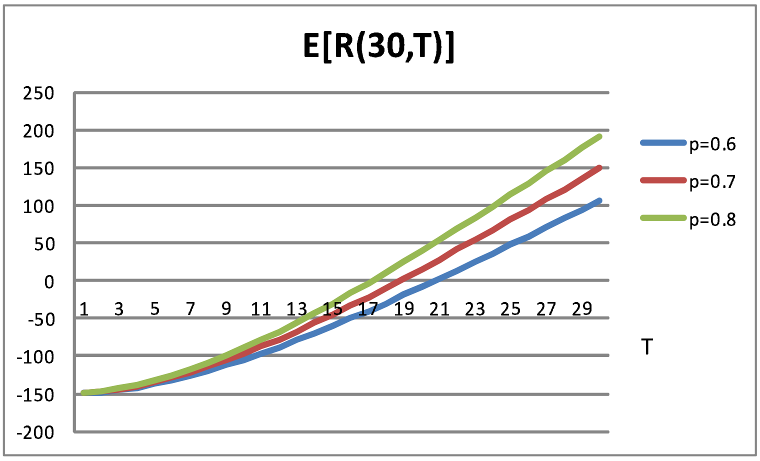

Figure 6.4.3: $\mathrm{E}[R(30, T)]$ as a Function of T

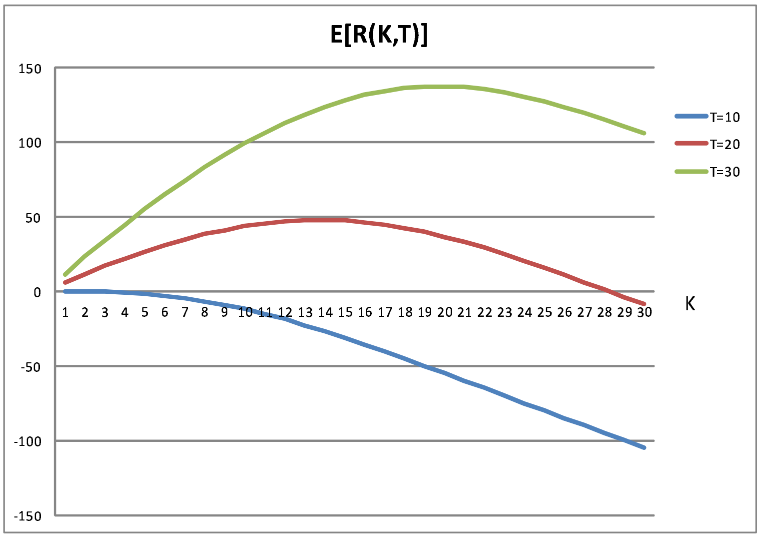

Figure 6.4.4: $\mathrm{E}[R(K,T)]$ as a Function of K for $p = 0.6$

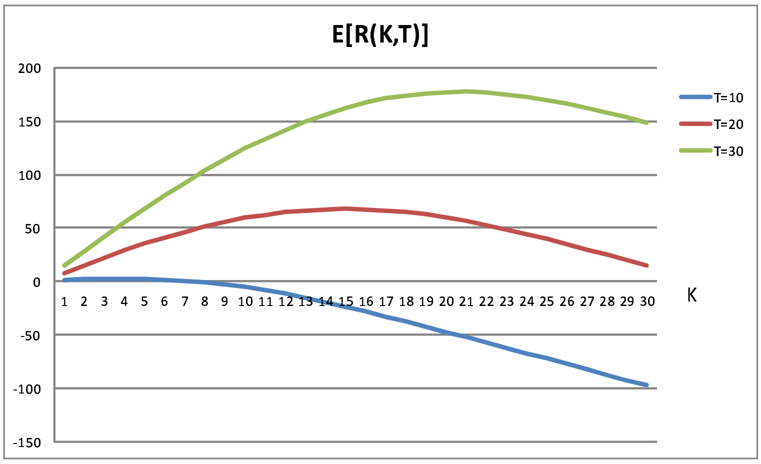

Figure 6.4.5: $\mathrm{E}[R(K,T)]$ as a Function of K for $p = 0.7$

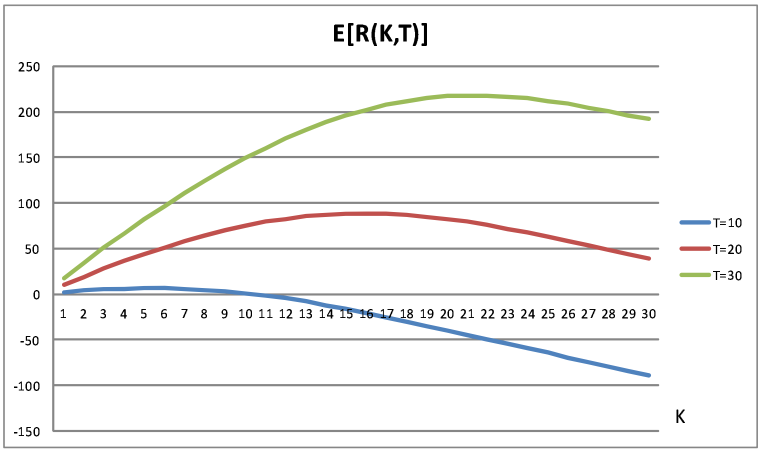

Figure 6.4.6: $\mathrm{E}[R(K,T)]$ as a Function of K for $p = 0.8$

Table 6.4.3: The Optimal Values K^*

T \ p	*0.6*	*0.7*	*0.8*
10	***2***	***5***	***6***
20	*15*	*16*	*17*
30	*20*	*22*	*22*

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Appendix A

On Non-Existence of Equilibrium of M Person Game

A.1 Proof of Preliminary Lemmas and Main Theorem

<u>Proof of Lemma 4.3.1</u> From (4.2.2), it can be readily seen that $AV_j(\underline{\pi}) \subset AV_j(\underline{U})$ for any $\underline{\pi} \in S$. Since $AV_j(\underline{U}) = \{i\}$ and $i \in AV_j(\underline{\pi})$, this then implies that $AV_j(\underline{\pi}) = \{i\}$. Hence, one has $AV_j(\underline{\pi}) \cap LE_i(\underline{\pi}) = \emptyset$ so that $W_{ij}(\underline{\pi}) = \{i\}$ from (4.2.5).

<u>Proof of Lemma 4.3.2</u> Since $\underline{\pi} \neq \underline{U}$, there exists at least one i satisfying $\pi_i < U$. Let j be such that $c_{ij} = \min_{n\in\mathcal{N}} c_{in}$. Then one has $c_{ij} < \pi_i$ from (4.2.1) so that $i \in AV_j(\underline{\pi})$. We consider the following two cases.

<u>Case1: $AV_j(\underline{\pi}) \cap LE_i(\underline{\pi}) = \emptyset$</u>

Since $i \in AV_j(\underline{\pi})$, one has $i \in W_{ij}(\underline{\pi})$ from (4.2.5) and hence $|W_{ij}(\underline{\pi})| \geq 1$.

<u>Case2: $AV_j(\underline{\pi}) \cap LE_i(\underline{\pi}) \neq \emptyset$</u>

Let i' be such that $\pi_{i'} = \min_{m\in AV_j(\underline{\pi})\cap LE_i(\underline{\pi})} \pi_m$. Then $AV_j(\underline{\pi}) \cap LE_{i'}(\underline{\pi}) = \emptyset$. One also sees that $i' \in AV_j(\underline{\pi}) \cap LE_i(\underline{\pi})$ implies $i' \in AV_j(\underline{\pi})$. These observations together with (4.2.5) imply that $i' \in W_{i'j}(\underline{\pi})$ and $|W_{i'j}(\underline{\pi})| \geq 1$.

<u>Proof of Lemma 4.3.3</u> We first prove part 1) by contraposition. Suppose $|W_{ij}(\pi_i^\sharp, \underline{\pi}^*_{\setminus i})| \geq 2$ for some j. Then from the definition of $W_{ij}(\underline{\pi})$ in (4.2.5), one has $|EQ_i(\pi_i^\sharp, \underline{\pi}^*_{\setminus i})| \geq 2$. From

(4.3.1) it is clear that

$$LA_i(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) = LA_i(\underline{\pi}^*) \quad . \tag{A.1.1}$$

Since $\Delta > 0$, it follows from (4.2.3) and (4.2.4) that $LE_i(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) = LE_i(\underline{\pi}^*) \cup (EQ_i(\underline{\pi}^*) \setminus \{i\})$. From this and (A.1.1), it is readily seen that

$$\begin{aligned} EQ_i(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) &= \mathcal{M} \setminus [LE_i(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) \cup LA_i(\pi_i^\sharp, \underline{\pi}^*_{\setminus i})] \\ &= \mathcal{M} \setminus [LE_i(\underline{\pi}^*) \cup (EQ_i(\underline{\pi}^*) \setminus \{i\}) \cup LA_i(\underline{\pi}^*)] \\ &= \mathcal{M} \setminus [(LE_i(\underline{\pi}^*) \cup EQ_i(\underline{\pi}^*) \cup LA_i(\underline{\pi}^*)) \setminus (\overline{LE_i}(\underline{\pi}^*) \cap \{i\} \cap \overline{LA_i}(\underline{\pi}^*))] \\ &= \mathcal{M} \setminus (\mathcal{M} \setminus \{i\}) = \{i\} \quad , \end{aligned}$$

which contradicts to $|EQ_i(\pi_i^\sharp, \underline{\pi}^*_{\setminus i})| \geq 2$.

For part 2), suppose $|W_{ij}(\underline{\pi}^*)| = 1$ and $|W_{ij}(\pi_i^\sharp, \underline{\pi}^*_{\setminus i})| \neq 1$ for some $j \in \mathcal{N}$. Then from part 1), one has $|W_{ij}(\pi_i^\sharp, \underline{\pi}^*_{\setminus i})| = 0$, and hence $W_{ij}(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) = \emptyset$. Accordingly from (4.2.5), one has either

$$AV_j(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) \cap LE_i(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) \neq \emptyset \quad \text{or} \quad i \in NA_j(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) \quad . \tag{A.1.2}$$

Similarly from (4.2.5), since $W_{ij}(\underline{\pi}^*) \neq \emptyset$ from the assumption, one has

$$AV_j(\underline{\pi}^*) \cap LE_i(\underline{\pi}^*) = \emptyset \ ; \quad \text{and} \tag{A.1.3}$$

$$i \in AV_j(\pi^*) \quad . \tag{A.1.4}$$

From (4.2.2) and (A.1.4), it is clear that

$$AV_j(\underline{\pi}^*) \subset AV_j(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) \quad . \tag{A.1.5}$$

Hence one has $i \in AV_j(\pi_i^\sharp, \underline{\pi}^*_{\setminus i})$ from (A.1.4) and (A.1.5). This, in turn, implies from (A.1.2) that

$$AV_j(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) \cap LE_i(\pi_i^\sharp, \underline{\pi}^*_{\setminus i}) \neq \emptyset. \tag{A.1.6}$$

It then follows from (A.1.3), (A.1.5) and (A.1.6) that

$$\begin{aligned} SE &\stackrel{\text{def}}{=} AV_j(\underline{\pi}^*) \cap [LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \setminus LE_i(\underline{\pi}^*)] \\ &= \{AV_j(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \cap LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)\} \setminus \{AV_j(\underline{\pi}^*) \cap LE_i(\underline{\pi}^*)\} \neq \emptyset \quad . \end{aligned} \tag{A.1.7}$$

Suppose $i' \in SE$. It is clear from (A.1.7), that $i' \in LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)$ and hence $i' \notin LA_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) = LA(\underline{\pi}^*)$ from (A.1.1). Since $i' \in SE$, one sees that $i' \notin LE_i(\underline{\pi}^*)$. Consequently, one has $i' \in EQ_i(\underline{\pi}^*)$. Thus $i' \in (EQ_i(\underline{\pi}^*) \cap AV_j(\underline{\pi}^*))$ so that $i' \in W_{ij}(\underline{\pi}^*)$. Since $i' \in LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)$, one has $i' \neq i$, so that $W_{ij}(\underline{\pi}^*) \supset \{i, i'\}$ and hence $|W_{ij}(\underline{\pi}^*)| \geq 2$, which contradicts to $|W_{ij}(\underline{\pi}^*)| = 1$, completing the proof.

Proof of Theorem 4.3.4 We first prove part 2) by contraposition. Suppose $\underline{\pi}^* \in \mathcal{NE}$ and $\underline{U} \neq \underline{\pi}^*$. From Lemma 4.3.1, there exists $\hat{i} \in \mathcal{M}$ and $\hat{j} \in \mathcal{N}$ such that $|W_{\hat{i}\hat{j}}(\underline{\pi}^*)| \geq 1$ and $\pi_{\hat{i}}^* < U$. We consider the following two cases.

Case1: $J_{\hat{i}j}(\underline{\pi}^*) = 0$ for all $j \in \mathcal{M}$

From the definition of $P_i(\underline{\pi})$ in (4.2.8), one sees that

$$P_{\hat{i}}(\underline{\pi}^*) = \sum_{j \in \mathcal{N}} D_j(\pi_{\hat{i}}^* - c_{\hat{i}j}) I_{\hat{i}j}(\underline{\pi}^*) \quad . \tag{A.1.8}$$

Let $\pi_{\hat{j}}^\sharp$ be as in (4.3.1). Then from 1) of Lemma 4.3.3, one has $J_{\hat{i}j}(\pi_{\hat{i}}^\sharp, \underline{\pi}_{\setminus\hat{i}}) = 0$ for all $j \in \mathcal{M}$. It then follows from this and (4.2.8) that

$$P_{\hat{i}}(\pi_{\hat{i}}^\sharp, \underline{\pi}_{\setminus\hat{i}}) = \sum_{j \in \mathcal{N}} D_j(\pi_{\hat{i}}^* + \Delta - c_{\hat{i}j}) I_{\hat{i}j}(\pi_{\hat{i}}^\sharp, \underline{\pi}_{\setminus\hat{i}}) \quad . \tag{A.1.9}$$

From 2) of lemma 4.3.3, $I_{\hat{i}j}(\underline{\pi}^*) = 1$ implies $I_{\hat{i}j}(\pi_{\hat{i}}^\sharp, \underline{\pi}_{\setminus\hat{i}}) = 1$ so that $I_{\hat{i}j}(\pi_{\hat{i}}^\sharp, \underline{\pi}_{\setminus\hat{i}}) - I_{\hat{i}j}(\underline{\pi}^*) \geq 0$ for all $j \in \mathcal{N}$. Since $|W_{\hat{i}\hat{j}}(\underline{\pi}^*)| \geq 1$ and $J_{\hat{i}\hat{j}}(\underline{\pi}^*) = 0$, it is clear that $I_{\hat{i}\hat{j}}(\underline{\pi}^*) = 1$. These observations together with (A.1.8) and (A.1.9) then yield that

$$
\begin{aligned}
&P_{\hat{i}}(\pi^{\sharp}_{\hat{i}}, \underline{\pi}_{\setminus\hat{i}}) - P_{\hat{i}}(\underline{\pi}^*) \\
&\quad = \sum_{j\in\mathcal{N}} \left[D_j(\pi^*_{\hat{i}} + \Delta - c_{\hat{i}j}) I_{\hat{i}j}(\pi^{\sharp}_{\hat{i}}, \underline{\pi}_{\setminus\hat{i}}) - D_j(\pi^*_{\hat{i}} - c_{\hat{i}j}) I_{\hat{i}j}(\underline{\pi}^*) \right] \\
&\qquad = \sum_{j\in\mathcal{N}} \Big[D_j(\pi^*_{\hat{i}} + \Delta - c_{\hat{i}j}) I_{\hat{i}j}(\underline{\pi}^*) - D_j(\pi^*_{\hat{i}} - c_{\hat{i}j}) I_{\hat{i}j}(\underline{\pi}^*) \\
&\qquad\qquad + D_j(\pi^*_{\hat{i}} + \Delta - c_{\hat{i}j}) \{ I_{\hat{i}j}(\pi^{\sharp}_{\hat{i}}, \underline{\pi}_{\setminus\hat{i}}) - I_{\hat{i}j}(\underline{\pi}^*) \} \Big] \\
&\qquad\qquad\qquad \geq \sum_{j\in\mathcal{N}} D_j \Delta I_{\hat{i}j}(\underline{\pi}^*) \ \geq \ D_{\hat{j}} \Delta I_{\hat{i}\hat{j}}(\underline{\pi}^*) \ > 0 \quad ,
\end{aligned}
$$

which contradicts to $\underline{\pi}^* \in \mathcal{NE}$.

Case2: $J_{\hat{i}j}(\underline{\pi}^*) = 1$ for some $j \in \mathcal{N}$

Since $\pi^*_{\hat{i}} > c_{\hat{i}n}$ for any customer n supplied by supplier $\hat{i}$, and $\pi^*_{\hat{i}} > \pi^*_m$ for any $m \in LE_{\hat{i}}(\underline{\pi}^*)$, one can choose $\Delta > 0$ sufficiently small so that $\pi^{\dagger}_{\hat{i}} = \pi^*_{\hat{i}} - \Delta$ satisfies

$$
\max \left[\max_{n\in\{n:(I_{\hat{i}n}(\underline{\pi}^*)=1)\vee(J_{\hat{i}n}(\underline{\pi}^*)=1)\}} \{c_{\hat{i}n}\} \quad , \qquad \max_{m\in LE_{\hat{i}}(\underline{\pi}^*)} \{\pi^*_m\} \right] < \pi^{\dagger}_{\hat{i}} \quad , \tag{A.1.10}
$$

where the second maximum in (A.1.10) is ignored if $LE_{\hat{i}}(\underline{\pi}^*) = \emptyset$. One then sees that $LE_{\hat{i}}(\pi^{\dagger}_{\hat{i}}, \underline{\pi}^*_{\setminus\hat{i}}) = LE_{\hat{i}}(\underline{\pi}^*)$, $EQ_{\hat{i}}(\pi^{\dagger}_{\hat{i}}, \underline{\pi}^*_{\setminus\hat{i}}) = \{\hat{i}\}$ and $LA_{\hat{i}}(\pi^{\dagger}_{\hat{i}}, \underline{\pi}^*_{\setminus\hat{i}}) = LA_{\hat{i}}(\underline{\pi}^*) \cup \big(EQ_{\hat{i}}(\underline{\pi}^*) \setminus \{\hat{i}\}\big)$. From (4.2.6) and (4.2.7), these observations imply that the following statements hold true for all $j \in \mathcal{N}$.

$$
a) \quad \text{If } \ I_{\hat{i}j}(\underline{\pi}^*) = 1 \qquad \text{then} \quad I_{\hat{i}j}(\pi^{\dagger}_{\hat{i}}, \underline{\pi}^*_{\setminus\hat{i}}) = 1 \tag{A.1.11}
$$

$$
b) \quad \text{If } \ J_{\hat{i}j}(\underline{\pi}^*) = 1 \qquad \text{then} \quad I_{\hat{i}j}(\pi^{\dagger}_{\hat{i}}, \underline{\pi}^*_{\setminus\hat{i}}) = 1 \text{ and } \quad J_{\hat{i}j}(\pi^{\dagger}_{\hat{i}}, \underline{\pi}^*_{\setminus\hat{i}}) = 0 \tag{A.1.12}
$$

$$
c) \quad \text{If } \ \big[I_{\hat{i}j}(\underline{\pi}^*) = 0 \wedge J_{\hat{i}j}(\underline{\pi}^*) = 0\big], \text{ then } \big[I_{\hat{i}j}(\pi^{\dagger}_{\hat{i}}, \underline{\pi}^*_{\setminus\hat{i}}) = 0 \wedge J_{\hat{i}j}(\pi^{\dagger}_{\hat{i}}, \underline{\pi}^*_{\setminus\hat{i}}) = 0\big] \tag{A.1.13}
$$

From the definition of $P_{\hat{i}}(\underline{\pi})$ in (4.2.8) together with (A.1.11), (A.1.12) and (A.1.13), one then sees that

$$\begin{aligned}
P_{\hat{i}}(\pi_{\hat{i}}^{\dagger}, \underline{\pi}_{\backslash\hat{i}}^{*}) - P_{\hat{i}}(\underline{\pi}^{*}) &= \sum_{j\in\mathcal{N}} D_j(\pi_{\hat{i}}^{\dagger} - c_{\hat{i}j}) I_{\hat{i}j}(\pi_{\hat{i}}^{\dagger}, \underline{\pi}_{\backslash\hat{i}}^{*}) - \sum_{j\in\mathcal{N}} D_j(\pi_{\hat{i}}^{\dagger} - c_{\hat{i}j}) \Big[I_{\hat{i}j}(\underline{\pi}^{*}) + \frac{J_{\hat{i}j}(\underline{\pi}^{*})}{|W_{\hat{i}j}(\underline{\pi}^{*})|} \Big] \\
&+ \sum_{j\in\mathcal{N}} D_j(\pi_{\hat{i}}^{\dagger} - c_{\hat{i}j}) \Big[I_{\hat{i}j}(\underline{\pi}^{*}) + \frac{J_{\hat{i}j}(\underline{\pi}^{*})}{|W_{\hat{i}j}(\underline{\pi}^{*})|} \Big] - \sum_{j\in\mathcal{N}} D_j(\pi_{\hat{i}}^{*} - c_{\hat{i}j}) \Big[I_{\hat{i}j}(\underline{\pi}^{*}) + \frac{J_{\hat{i}j}(\underline{\pi}^{*})}{|W_{\hat{i}j}(\underline{\pi}^{*})|} \Big] \\
&= \sum_{j\in\mathcal{N}} D_j(\pi_{\hat{i}}^{\dagger} - c_{\hat{i}j}) \Big[1 - \frac{1}{|W_{\hat{i}j}(\underline{\pi}^{*})|} \Big] J_{\hat{i}j}(\underline{\pi}^{*}) - \Delta \sum_{j\in\mathcal{N}} D_j \Big[I_{\hat{i}j}(\underline{\pi}^{*}) + \frac{J_{\hat{i}j}(\underline{\pi}^{*})}{|W_{\hat{i}j}(\underline{\pi}^{*})|} \Big] \\
&= \sum_{j\in\mathcal{N}} D_j(\pi_{\hat{i}}^{*} - c_{\hat{i}j}) \Big[1 - \frac{1}{|W_{\hat{i}j}(\underline{\pi}^{*})|} \Big] J_{\hat{i}j}(\underline{\pi}^{*}) - \Delta \sum_{j\in\mathcal{N}} D_j \{ I_{\hat{i}j}(\underline{\pi}^{*}) + J_{\hat{i}j}(\underline{\pi}^{*}) \} \\
&\geq D_{\hat{j}}(\pi_{\hat{i}}^{*} - c_{\hat{i}\hat{j}}) \Big[1 - \frac{1}{|W_{\hat{i}\hat{j}}(\underline{\pi}^{*})|} \Big] - \Delta \sum_{j\in\mathcal{N}} D_j \{ I_{\hat{i}j}(\underline{\pi}^{*}) + J_{\hat{i}j}(\underline{\pi}^{*}) \} \quad .
\end{aligned}$$

Since the first component in the last term is positive, one can choose Δ sufficiently small so that $P_{\hat{i}}(\pi_{\hat{i}}^{\dagger}, \underline{\pi}_{\backslash\hat{i}}^{*}) > P_{\hat{i}}(\underline{\pi}^{*})$, which contradicts to $\underline{\pi}^{*} \in \mathcal{NE}$, completing the proof for part 2).

We next prove "if part" of part 1). If $|AV_j(\underline{U})| \leq 1$ for all $j \in \mathcal{N}$, then from Lemma 4.3.1, one has $|W_{ij}(\underline{\pi})| \leq 1$ for all $\underline{\pi} \in S$ and $i \in \mathcal{M}$. Hence, for all $\underline{\pi} \in S$ and $i \in \mathcal{M}$, one has $P_i(\underline{\pi}) = \sum_{j\in\mathcal{N}} D_j(\pi_i - c_{ij}) I_{ij}(\underline{\pi})$. It then follows that

$$P_i(\underline{U}) - P_i(\underline{\pi}) = \sum_{j\in\mathcal{N}} D_j(U - c_{ij}) \{ I_{ij}(\underline{U}) - I_{ij}(\underline{\pi}) \} + \sum_{j\in\mathcal{N}} D_j(U - \pi_i) I_{ij}(\underline{\pi}) \ . \tag{A.1.14}$$

If $U \leq c_{ij}$ then $W_{ij}(\underline{\pi}) = \emptyset$, so that $I_{ij}(\underline{\pi}) = 0$ and hence $I_{ij}(\underline{\pi}) = I_{ij}(U)$ for all $\underline{\pi} \in S$. If $U > c_{ij}$ (and hence $AV_j(\underline{U}) = \{i\}$ and $I_{ij}(\underline{\pi}) = 1$) then, from Lemma 4.3.1, one has $AV_j(\underline{\pi}) = \{i\}$ so that $I_{ij}(\underline{\pi}) = 1$ for any price vector $\underline{\pi}$ with $\pi_i > c_{ij}$. Hence $I_{ij}(\underline{U}) = I_{ij}(\underline{\pi})$ for all $\underline{\pi} \in S$. These observation then imply that the payoff difference in (A.1.14) is non-negative for all $\underline{\pi} \in S$. It then follows that $U \in B_i(\underline{U}_{\backslash i})$ for all $i \in \mathcal{M}$. Hence one has $\underline{U} \in \mathcal{NE}$, proving "if part".

For "only if part", suppose $\mathcal{NE} \neq \emptyset$ and $|AV_j(\underline{U})| \geq 2$ for some $\hat{j} \in \mathcal{N}$. From part 2) of this theorem one has $\mathcal{NE} = \{\underline{U}\}$. To emphasize this, we write $\underline{\pi}^{*} = \underline{U}$. Let $\hat{i}, \hat{i}' \in AV_{\hat{j}}(\underline{\pi}^{*})$. Since $LE_{\hat{i}}(\underline{\pi}^{*}) = \emptyset$ from (4.2.3), the definition of $W_{ij}(\underline{\pi})$ in (4.2.5) implies $\hat{i} \in W_{\hat{i}\hat{j}}(\underline{\pi}^{*})$. Since

$\pi_{\hat{i}} = \pi_{\hat{i}'} = U$, it is clear that $\hat{i}' \in EQ_{\hat{i}}(\underline{\pi}^*)$ thus $\hat{i}' \in W_{\hat{i}\hat{j}}(\underline{\pi}^*)$, so that $J_{\hat{i}\hat{j}}(\underline{\pi}^*) = 1$. Let $\pi_i^\dagger = \pi_i^* - \Delta$ for sufficiently small Δ as in (A.1.10). Similarly as in the proof of Case2 of part 2), statements (A.1.11), (A.1.12) and (A.1.13) hold true. These together with the definition of $P_i(\underline{\pi})$ in (4.2.8) imply that

$$\begin{aligned} & P_{\hat{i}}(\pi_{\hat{i}}^\dagger, \underline{\pi}_{\backslash\hat{i}}^*) - P_{\hat{i}}(\underline{\pi}^*) \\ & \quad \geq D_{\hat{j}}(\pi_{\hat{i}}^* - c_{\hat{i}\hat{j}})\Big[1 - \frac{1}{|W_{\hat{i}\hat{j}}(\underline{\pi}^*)|}\Big] J_{\hat{i}\hat{j}}(\underline{\pi}^*) - \Delta \sum_{j \in \mathcal{N}} D_j \Big[I_{\hat{i}j}(\underline{\pi}^*) + J_{\hat{i}j}(\underline{\pi}^*)\Big] \quad . \end{aligned} \tag{A.1.15}$$

Since the first component in the last term in (A.1.15) is positive, one can choose Δ sufficiently small so that $P_{\hat{i}}(\pi_{\hat{i}}^\dagger, \underline{\pi}_{\backslash i}^*) > P_{\hat{i}}(\underline{\pi}^*)$, which contradicts $\underline{\pi}^* \in \mathcal{NE}$, proving “only if part” of part 2) .

Appendix B

Structural Analysis of Two Person Game

B.1 Proof of Theorem 5.3.2 and Theorem 5.3.3

Before the proof of Theorem 5.3.2 and 5.3.3, the following matrices are defined and several preliminary lemmas are given. We note that $\delta_{\{ST\}} = 1$ if the statement ST holds and $\delta_{\{ST\}} = 0$ else.

Definition B.1

$\underline{\underline{I}} = [\delta_{\{m=n\}}]_{m,n\in\mathcal{L}\setminus\{L\}} \in \mathcal{R}^{(L-1)\times(L-1)}$

$\underline{\underline{A}}_D = [\delta_{\{m=n\}}a_{m+1}]_{m,n\in\mathcal{L}\setminus\{L\}} \in \mathcal{R}^{(L-1)\times(L-1)}$

$\underline{\underline{L}} = [\delta_{\{m<n\}}]_{m,n\in\mathcal{L}\setminus\{L\}} \in \mathcal{R}^{(L-1)\times(L-1)}$

$\underline{\underline{L}}_1 = [\delta_{\{m+1=n\}}]_{m,n\in\mathcal{L}\setminus\{L\}} \in \mathcal{R}^{(L-1)\times(L-1)}$

$\underline{\underline{B}} = \underline{\underline{I}} + \underline{\underline{L}} \in \mathcal{R}^{(L-1)\times(L-1)}$

$\underline{\underline{C}} = \underline{\underline{I}} + 2\underline{\underline{L}} \in \mathcal{R}^{(L-1)\times(L-1)}$

Lemma B.2

$$a) \quad \underline{\underline{B}}^{-1}\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} = \underline{w}(\Delta, \frac{1}{a_L}) \; ; \quad \text{and}$$

$$b) \quad \underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{1}_{L-1} = \underline{w}(2,1) \quad .$$

Proof

We first note that $\underline{\underline{L}} - \underline{\underline{L}}_1 = \underline{\underline{L}}_1\underline{\underline{L}}$ so that $(\underline{\underline{I}} - \underline{\underline{L}}_1)\underline{\underline{B}} = \underline{\underline{I}} - \underline{\underline{L}}_1 + \underline{\underline{L}} - \underline{\underline{L}}_1\underline{\underline{L}} = \underline{\underline{I}}$, and hence $\underline{\underline{B}}^{-1} = \underline{\underline{I}} - \underline{\underline{L}}_1$. From (5.3.6) and Defition B.1, one then sees that $\underline{\underline{B}}^{-1}\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} = (\underline{\underline{I}} - \underline{\underline{L}}_1)\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} = \underline{\underline{A}}_D^{-1}\underline{1}_{L-1} - \underline{\underline{L}}_1\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} = \Delta\underline{1}_{L-1} + \left(\frac{1}{a_L} - \Delta\right)\underline{e}_{L-1}$, where Δ is as in (5.3.4), proving a). For part b), since $\underline{\underline{B}}^{-1} = \underline{\underline{I}} - \underline{\underline{L}}_1$ and $(\underline{\underline{I}} - \underline{\underline{L}}_1)\underline{\underline{L}} = \underline{\underline{L}}_1$, it can be seen that $\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{1}_{L-1} = (\underline{\underline{I}} - \underline{\underline{L}}_1)(\underline{\underline{I}} + 2\underline{\underline{L}})\underline{1}_{L-1} = \{(\underline{\underline{I}} - \underline{\underline{L}}_1) + (2\underline{\underline{L}} - 2\underline{\underline{L}}_1\underline{\underline{L}})\}\underline{1}_{L-1} = \underline{\underline{I}}\,\underline{1}_{L-1} + \underline{\underline{L}}_1\underline{1}_{L-1} = \underline{w}(1,1) + \underline{w}(1,0) = \underline{w}(2,1)$ where $\underline{\underline{I}}\,\underline{1}_{L-1} = \underline{w}(1,1)$ and $\underline{\underline{L}}_1\underline{1}_{L-1} = \underline{w}(1,0)$ are employed to yield the last eauality, proving the lemma.

<u>Lemma B.3</u>

Let $\underline{\underline{H}}$ be as in (5.3.9) and define $\underline{v}_L^T = [v_1, \cdots, v_L]$ as in (5.3.7). Then one has

$a)[\underline{\underline{H}}]_{1,m} = 2a_1$ for $m \in \mathcal{L}$;

$b)[\underline{\underline{H}}]_{n,1} = a_1 + a_n$ for $n \in \mathcal{L} \setminus \{1\}$;

$c)[\underline{\underline{H}}]_{m,n} = [\underline{\underline{A}}_D\underline{\underline{C}}]_{m-1,n-1}$ for $m, n \in \mathcal{L} \setminus \{1\}$;

$$d)\underline{\underline{H}} = \begin{bmatrix} 2a_1 & 2a_1\underline{1}_{L-1}^T \\ (a_1\underline{\underline{I}} + \underline{\underline{A}}_D)\underline{1}_{L-1} & \underline{\underline{A}}_D\underline{\underline{C}} \end{bmatrix};$$

$$e)\underline{\underline{H}}\begin{bmatrix} x \\ y\underline{1}_{L-1} \end{bmatrix} = \begin{bmatrix} 2a_1 \\ (y\underline{\underline{A}}_D\underline{\underline{C}} + xa_1\underline{\underline{I}} + x\underline{\underline{A}}_D)\underline{1}_{L-1} \end{bmatrix}$$

where $0 < x < 1$, *and* $y = (1-x)/(L-1)r$.

Proof

In what follows, since $\underline{\underline{H}} = \underline{\underline{H}}_1$ as in (5.3.9), any reference to (5.2.1) assumes $i = 1$. We first note from (5.3.1) and (5.3.7) that $v_1 = \frac{a_1}{D} + c_{mid} = \frac{c_{high} - c_{low}}{2} + \frac{c_{high} + c_{low}}{2} = c_{high}$. Hence from (5.2.1) and (5.3.1), one has $[\underline{\underline{H}}]_{1,m} = h_1(v_1, v_m) = h_1(c_{high}, v_m) = (c_{high} - c_{low})D = 2a_1$, proving a). For part b), one sees from (5.2.1) that $[\underline{\underline{H}}]_{n,1} = h_1(v_n, v_1) = h_1(v_n, c_{high}) = (v_n - c_{low})D$. Substituting $v_n = \frac{a_n}{D} + c_{mid}$ from (5.3.7) into the last term and using (5.3.1), we obtain $(v_n - c_{low})D = a_n + \frac{c_{high} - c_{low}}{2}D = a_n + a_1$. In order to prove part c), we consider the following three cases:

Case1: $1 < m < n \leq L$

For this case, one has $v_m < v_n$ from (5.3.6) and (5.3.7) so that it follows from (5.2.1) that $[\underline{\underline{H}}]_{m,n} = h(v_m, v_n) = 2(v_m - c_{mid})D = 2(\frac{a_m}{D} + c_{mid} - c_{mid})D = 2a_m$.

Case2: $m = n \leq L$

Similarly, for $m = n$, one has $[\underline{\underline{H}}]_{m,n} = h(v_m, v_n) = (v_m - c_{mid})D = a_m$ for $m \in \mathcal{L} \setminus \{1\}$.

Case3: $L \geq m > n > 1$

In this case, one has $v_m > v_n$ and from (5.2.1) $[\underline{\underline{H}}]_{m,n} = 0$.

These observations imply that that

$$\underline{\underline{A}}_D \underline{\underline{C}} = \begin{bmatrix} a_2 & 2a_2 & 2a_2 & \cdots & 2a_2 \\ & a_3 & 2a_3 & \cdots & 2a_3 \\ & & a_4 & \cdots & 2a_4 \\ & \underline{\underline{0}} & & \ddots & \vdots \\ & & & & a_L \end{bmatrix}$$

and part c) follows. Part d) is immediate from a), b), and c). Finally we prove part e). Using the result of d), one sees that $\underline{\underline{H}} \begin{bmatrix} x \\ y\underline{1}_{L-1} \end{bmatrix} = \begin{bmatrix} 2a_1 & 2a_1\underline{1}^T_{L-1} \\ (a_1\underline{\underline{I}} + \underline{\underline{A}}_D)\underline{1}_{L-1} & \underline{\underline{A}}_D\underline{\underline{C}} \end{bmatrix} \begin{bmatrix} x \\ y\underline{1}_{L-1} \end{bmatrix}$
$= \begin{bmatrix} 2a_1\{x + y(L-1)\} \\ \{a_1 x\underline{\underline{I}} + x\underline{\underline{A}}_D + y\underline{\underline{A}}_D\underline{\underline{C}}\}\underline{1}_{L-1} \end{bmatrix}$
$= \begin{bmatrix} 2a_1 \\ \{y\underline{\underline{A}}_D\underline{\underline{C}} + xa_1\underline{\underline{I}} + x\underline{\underline{A}}_D\}\underline{1}_{L-1} \end{bmatrix}$.

<u>Lemma B.4</u>:

a)$\alpha_3 = a_1(2 - \alpha_3)\frac{1}{a_L}$; *b)*$\alpha_4 = a_1(2 - \alpha_3)\Delta$.

Proof

By the definition of α_3, one sees that $\alpha_3(1 + \frac{a_1}{a_L}) = 2\frac{a_1}{a_L}$, so that $\alpha_3 = 2\frac{a_1}{a_L} - \frac{a_1}{a_L}\alpha_3 = a_1(2 - \alpha_3)\frac{1}{a_L}$, proving a). For part b), we first note that $2 - \alpha_3 = 2 - \frac{2\frac{a_1}{a_L}}{1+\frac{a_1}{a_L}} = \frac{2}{1+\frac{a_1}{a_L}}$. Hence from Definition 5.3.1, one sees that $\alpha_4 = a_1\frac{2}{1+\frac{a_1}{a_L}}\Delta = a_1(2 - \alpha_3)\Delta$.

<u>Lemma B.5</u>

If L is even, then for any $0 < x < 1$ and $y = 2(1-x)/(L-2)$, one has

$$\underline{\underline{H}} \begin{bmatrix} x \\ y\underline{f} \end{bmatrix} = \begin{bmatrix} 2a_1 \\ y\underline{\underline{A}}_D\underline{\underline{C}}\,\underline{f} + xa_1\underline{1} + x\underline{\underline{A}}_D\underline{1} \end{bmatrix}.$$

Proof

From Lemma B.3 d), one sees $\underline{\underline{H}}\begin{bmatrix} x \\ y\underline{f} \end{bmatrix}=\begin{bmatrix} 2a_1 & 2a_1\underline{1}_{L-1}^T \\ (a_1\underline{\underline{I}}+\underline{\underline{A}}_D)\underline{1}_{L-1} & \underline{\underline{A}}_D\underline{\underline{C}} \end{bmatrix}$
$\begin{bmatrix} x \\ y\underline{f} \end{bmatrix}=\begin{bmatrix} 2a_1(x+y\frac{L-2}{2}) \\ a_1x\underline{\underline{I}}\,\underline{1}_{L-1}+x\underline{\underline{A}}_D\underline{1}_{L-1}+y\underline{\underline{A}}_D\underline{\underline{C}}\,\underline{f} \end{bmatrix}$
$=\begin{bmatrix} 2a_1 \\ y\underline{\underline{A}}_D\underline{\underline{C}}\,\underline{f}+a_1x\underline{1}_{L-1}+x\underline{\underline{A}}_D\underline{1}_{L-1} \end{bmatrix}.$

Lemma B.6:

Let $\underline{\underline{H}}, \underline{v}_L$ and $\underline{w}(x,y)$ be as in (5.3.9),(5.3.7) and Definition B.1 respectively. Then for any $0<y<1$ and $x=(1-y)/(L-2)$, one has

$$\underline{\underline{H}}\begin{bmatrix} 0 \\ \underline{w}(x,y) \end{bmatrix}=\begin{bmatrix} 2a_1 \\ \underline{\underline{A}}_D\underline{\underline{C}}\,\underline{w}(x,y) \end{bmatrix} \quad .$$

Proof

From Lemma B.3 d), one sees that $\underline{\underline{H}}\begin{bmatrix} 0 \\ \underline{w}(x,y) \end{bmatrix}=\begin{bmatrix} 2a_1\underline{1}_{L-1}^T\underline{w}(x,y) \\ \underline{\underline{A}}_D\underline{\underline{C}}\,\underline{w}(x,y) \end{bmatrix}=\begin{bmatrix} 2a_1 \\ \underline{\underline{A}}_D\underline{\underline{C}}\,\underline{w}(x,y) \end{bmatrix}$ where $\underline{1}_{L-1}^T\underline{w}(x,y)=(L-2)x+y=1$ is employed to yield the last equality.

Lemma B.7:

Let $\underline{f}$ be as in Definition 5.3.1. If L is even, then one has a) $\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{f}=\underline{w}(1,0)$ and b) $\underline{\underline{B}}^{-1}\underline{1}=\underline{w}(0,1)$.

Proof

We first note that $(\underline{\underline{I}}-\underline{\underline{L}}_1)\underline{\underline{L}}=\underline{\underline{L}}_1$ and $\underline{\underline{B}}^{-1}=\underline{\underline{I}}-\underline{\underline{L}}_1$ so that $\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{f}=(\underline{\underline{I}}-\underline{\underline{L}}_1)(\underline{\underline{I}}+2\underline{\underline{L}})\underline{f}=(\underline{\underline{I}}-\underline{\underline{L}}_1+2\underline{\underline{L}}_1)\underline{f}=(\underline{\underline{I}}+\underline{\underline{L}}_1)\underline{f}=\underline{w}(1,0)$, proving part a). For part b), one sees that $\underline{\underline{B}}^{-1}\underline{1}_{L-1}=(\underline{\underline{I}}-\underline{\underline{L}}_1)\underline{1}_{L-1}=\underline{1}_{L-1}-\underline{w}(1,0)=\underline{w}(0,1)$.

We are now in a position to prove Theorem 5.3.2 and 5.3.3.

Proof of Theorem 5.3.2:

Since $L>2$, from the definition of $\underline{q}^*$, it is clear $\underline{q}^*\in DRV(\underline{v}_L)$. In order to prove $(\underline{q}^*,\underline{q}^*)\in\mathcal{NE}(\underline{v}_L)$, from (5.3.8), all we need to show is that $V_1(\underline{e}_m,\underline{q}^*)\le V_1(\underline{q}^*,\underline{q}^*)$ and $V_2(\underline{q}^*,\underline{e}_m)\le V_2(\underline{q}_1^*,\underline{q}^*)$ hold for all $m\in\mathcal{L}$. From Definition B.1, one sees that $\underline{w}(x,y)$ is linear in (x,y).

It is clear from Definition 5.3.1 that a) $2\alpha_2 + a_1(\alpha_1 - 2)\Delta = 0$ and b) $\alpha_2 + a_1(\alpha_1 - 2)\frac{1}{a_L} + \alpha_1 = 0$. This then implies that $\alpha_2\underline{w}(2,1) + a_1(\alpha_1 - 2)\underline{w}(\Delta, \frac{1}{a_L}) + \alpha_1\underline{w}(0,1) = \underline{w}\big[2\alpha_2 + a_1(\alpha_1 - 2)\Delta, \alpha_2 + a_1(\alpha_1 - 2)\frac{1}{a_L} + \alpha_1\big] = \underline{w}(0,0) = \underline{0}$. With $\underline{w}(2,1)$ and $\underline{w}(\Delta, \frac{1}{a_L})$ in the above equation substituted by Lemma B.2 a) and b) respectively, one sees that $\alpha_2\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{1}_{L-1} + a_1(\alpha_1 - 2)\underline{\underline{B}}^{-1}\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} + \alpha_1\underline{w}(0,1) = \underline{0}$. Multiplying $\underline{\underline{A}}_D\underline{\underline{B}}$ from left, this then leads to $\alpha_2\underline{\underline{A}}_D\underline{\underline{B}}\,\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{1}_{L-1} + a_1(\alpha_1 - 2)\underline{1}_{L-1} + \alpha_1\underline{\underline{A}}_D\underline{\underline{B}}\,\underline{w}(0,1) = \underline{0}$ i.e.

$$\big[\alpha_2\underline{\underline{A}}_D\underline{\underline{C}} + a_1\alpha_1\underline{\underline{I}} + \alpha_1\underline{\underline{A}}_D\big]\underline{1}_{L-1} = 2a_1\underline{1}_{L-1}, \tag{B.1.1}$$

where $\underline{\underline{B}}\,\underline{w}(0,1) = \underline{1}_{L-1}$ is employed to yield (B.1.1). On the other hand, from Lemma B.3 d), one sees that $\underline{\underline{H}}\,\underline{q}^* = \begin{bmatrix} 2a_1 \\ (\alpha_2\underline{\underline{A}}_D\underline{\underline{C}} + \alpha_1 a_1\underline{\underline{I}} + \alpha_1\underline{\underline{A}}_D)\underline{1}_{L-1} \end{bmatrix}$. It then follows from this and (B.1.1) that $\underline{\underline{H}}\,\underline{q}^* = 2a_1\underline{1}_L$. This in turn implies that $V_1(\underline{e}_m, \underline{q}^*) = 2a_1 = V_1(\underline{q}^*, \underline{q}^*)$ and $V_2(\underline{q}^*, \underline{e}_m) = 2a_1 = V_2(\underline{q}^*, \underline{q}^*)$ hold for all $m \in \mathcal{L}$, completing the proof.

Proof of Theorem 5.3.3:

Since $L > \frac{a_L}{2a_1} + 1$, from the definition of $\underline{q}^\sharp$ and $\underline{q}^\dagger$, it is clear that $\underline{q}^\sharp, \underline{q}^\dagger \in DRV(\underline{v}_L)$. We next show that $V_1(\underline{e}_m, \underline{q}^\dagger) \le V_1(\underline{q}^\sharp, \underline{q}^\dagger)$ and $V_2(\underline{q}^\sharp, \underline{e}_m) \le V_2(\underline{q}^\sharp, \underline{q}^\dagger)$ hold for all $m \in \mathcal{L}$. From Lemma B.7, one has $\alpha_4\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{f} + \alpha_3\underline{\underline{B}}^{-1}\underline{1}_{L-1} = \alpha_4\underline{w}(1,0) + \alpha_3\underline{w}(0,1) = \underline{w}(\alpha_4, \alpha_3)$. By Lemma B.4, this then leads to $\alpha_4\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{f} + \alpha_3\underline{\underline{B}}^{-1}\underline{1}_{L-1} = a_1(2 - \alpha_3)\underline{w}(\Delta, \frac{1}{a_L}) = a_1(2 - \alpha_3)\underline{\underline{B}}^{-1}\underline{\underline{A}}_D^{-1}\underline{1}_{L-1}$ where Lemma B.2 a) is employed to yield the last equality. By multiplying $\underline{\underline{A}}_D\underline{\underline{B}}$ from left to the above equation, it follows that $\alpha_4\underline{\underline{A}}_D\underline{\underline{C}}\,\underline{f} + \alpha_3\underline{\underline{A}}_D\,\underline{1}_{L-1} = a_1(2 - \alpha_3)\underline{1}_{L-1}$, and one has

$$\alpha_4\underline{\underline{A}}_D\underline{\underline{C}}\,\underline{f} + \alpha_3\underline{\underline{A}}_D\underline{1}_{L-1} + \alpha_3 a_1\underline{1}_{L-1} = 2a_1\underline{1}_{L-1}. \tag{B.1.2}$$

Let $x = \frac{4\alpha_3}{4-\alpha_4}$ and $y = \frac{4\alpha_4}{4-\alpha_4}$. One sees that $(L-2)(4-\alpha_4)y = (L - \frac{3}{2} - \frac{1}{2})4\frac{2\Delta}{\frac{1}{a_1}+\frac{1}{a_L}} = 8 - 8\frac{\frac{2}{a_L}}{\frac{1}{a_1}+\frac{1}{a_L}} - 2\alpha_4 = 8 - 8\alpha_3 - 2\alpha_4 = 2(4-\alpha_4)(1-x)$, so that $y = \frac{2(1-x)}{L-2}$. Since $\underline{q}^{\sharp T} \in DRV(\underline{v}_L)$ and the first component of $\underline{q}^{\sharp T}$ is x, one has $0 < x < 1$. Applying these x and y to Lemma B.5 and using (B.1.2), one sees that $\underline{\underline{H}}\,\underline{q}^\sharp = \begin{bmatrix} 2a_1 \\ \frac{4}{4-\alpha_4}2a_1\underline{1}_{L-1} \end{bmatrix}$. This in turn implies that

$V_2(\underline{q}^\sharp, \underline{e}_m) = \underline{e}_m \,\underline{\underline{H}}\,\underline{q}^\sharp \le \frac{4}{4-\alpha_4} 2a_1 = V_2(\underline{q}^\sharp, \underline{q}^\dagger)$ for all $m \in \mathcal{L}$.

We also need to show $V_1(\underline{e}_m, \underline{q}^\dagger) \le V_1(\underline{q}^\sharp, \underline{q}^\dagger)$ for all $m \in \mathcal{L}$. From Definition 5.3.1, one sees that

$$\begin{aligned} & \alpha_5 \underline{w}(2,1) + \underline{w}(0, 2(\alpha_6 - \alpha_5)) \\ = \;& \underline{w}(2\alpha_5, 2\alpha_6 - \alpha_5) = 2a_1 \underline{w}(\Delta, \frac{1}{a_L}). \end{aligned} \tag{B.1.3}$$

Since $\underline{\underline{B}}^{-1}(\underline{\underline{C}} + \underline{\underline{I}}) = (\underline{\underline{I}} + \underline{\underline{L}})^{-1}(\underline{\underline{I}} + 2\underline{\underline{L}} + \underline{\underline{I}}) = 2\underline{\underline{I}}$, Lemma B.2 a) b) and (B.1.3) lead to

$$\begin{aligned} & \alpha_5 \underline{\underline{B}}^{-1} \underline{\underline{C}}\, \underline{1}_{L-1} + \underline{\underline{B}}^{-1}\, (\underline{\underline{C}} + \underline{\underline{I}}) \underline{w}(0, \alpha_6 - \alpha_5) \\ & \qquad = 2a_1 \underline{\underline{B}}^{-1} \underline{\underline{A}}_D^{-1}\, \underline{1}_{L-1} \,. \end{aligned} \tag{B.1.4}$$

Multiplying $\underline{\underline{A}}_D \underline{\underline{B}}$ from left in (B.1.4), one obtains $\alpha_5\, \underline{\underline{A}}_D \underline{\underline{C}}\, \underline{1}_{L-1} + \underline{\underline{A}}_D\, (\underline{\underline{C}} + \underline{\underline{I}})\, \underline{w}(0, \alpha_6 - \alpha_5) = 2a_1 \underline{1}_{L-1}$. From the linearity of $\underline{w}(x, y)$ and the definition of $\underline{\underline{A}}_D$ in Definition B.1, this then leads to $\underline{\underline{A}}_D \underline{\underline{C}}\, \underline{w}(\alpha_5, \alpha_6) + \underline{w}(0, a_L(\alpha_6 - \alpha_5)) = \underline{\underline{A}}_D\, \underline{\underline{C}}\, \underline{w}(\alpha_5, \alpha_5) + \underline{\underline{A}}_D\, \underline{\underline{C}}\, \underline{w}(0, \alpha_6 - \alpha_5) + \underline{w}(0, a_L(\alpha_6 - \alpha_5)) = \underline{\underline{A}}_D\, \underline{\underline{C}}\, \underline{w}(\alpha_5, \alpha_5) + \underline{\underline{A}}_D\, \underline{\underline{C}}\, \underline{w}(0, \alpha_6 - \alpha_5) + \underline{\underline{A}}_D\, \underline{w}(0, (\alpha_6 - \alpha_5)) = \alpha_5\, \underline{\underline{A}}_D \underline{\underline{C}}\, \underline{1}_{L-1} + \underline{\underline{A}}_D\, (\underline{\underline{C}} + \underline{\underline{I}})\, \underline{w}(0, \alpha_6 - \alpha_5) = 2a_1 \underline{1}_{L-1}$, that is,

$$\underline{\underline{A}}_D\, \underline{\underline{C}}\, \underline{w}(\alpha_5, \alpha_6) = 2a_1 \underline{1}_{L-1} - \underline{w}(0, a_L(\alpha_6 - \alpha_5)). \tag{B.1.5}$$

Let $x = \alpha_5$ and $y = \alpha_6$ so that $x(L-2) = a_1\Delta(L-2) = a_1\Delta(L - \frac{3}{2} - \frac{1}{2}) = (1 - \frac{a_1}{a_L}) - a_1\frac{\Delta}{2} = 1 - \alpha_6 = 1 - y$, and therefore $x = (1-y)/(L-2)$. From the definition of α_6 in Definition 5.3.1 a) and the condition $L > 2$, one has $y = \alpha_6 = a_1(\frac{1}{a_L} + \frac{\Delta}{2}) = \frac{a_1}{a_L} + \frac{a_1}{2}\frac{(1/a_1)-(1/a_L)}{L-(3/2)} < \frac{a_1}{a_L} + \frac{1}{2}\frac{1-(a_1/a_L)}{2-(3/2)} = 1$. Hence with x and y above, Lemma B.6 can be applied, yielding $\underline{\underline{H}}\,\underline{q}^\dagger = \underline{\underline{H}} \begin{bmatrix} 0 \\ \underline{w}(\alpha_5, \alpha_6) \end{bmatrix} = \begin{bmatrix} 2a_1 \\ \underline{\underline{A}}_D \underline{\underline{C}}\, \underline{w}(\alpha_5, \alpha_6) \end{bmatrix}$ $= \begin{bmatrix} 2a_1 \\ 2a_1 \underline{1}_{L-1} - \underline{w}(0, a_L(\alpha_6 - \alpha_5)) \end{bmatrix}$. It should be noted that from the condition $L > \frac{a_L}{2a_1} + 1$, one has $\alpha_6 - \alpha_5 = \frac{a_1}{a_L} - \frac{a_1\Delta}{2} = \frac{a_1}{a_L} - \frac{1-\frac{a_1}{a_L}}{2(L-\frac{3}{2})} > \frac{a_1}{a_L} - \frac{1-\frac{a_1}{a_L}}{2(\frac{a_L}{2a_1}+1-\frac{3}{2})} = \frac{a_1}{a_L} - \frac{1-\frac{a_1}{a_L}}{\frac{a_L}{a_1}-1} = 0$, so that $V_1(\underline{e}_m, \underline{q}^\dagger) \le 2a_1 = V_1(\underline{q}^\sharp, \underline{q}^\dagger)$ for all $m \in \mathcal{L}$, where $(\underline{q}^\sharp)_L = 0$ if L is even is employed to yield the last equality.

Before the proof of Theorem 5.4.2, following lemma is needed.

Lemma C.8:

$a) F^*_\infty(v_{L:m}) < r^*_{L:m+1}\,, m = 1, \cdots, L-1$

$b) r^*_{L:m} < F^*_\infty(v_{L:m})\,, m = 1, \cdots, L-1$

$c) F^*_\infty(v_{L:m}) < r^\sharp_{L:m}\,, m = 1, 3, 5, \cdots, L-1$

$d) r^\sharp_{L:m} < F^*_\infty(v_{L:m+2})\,, m = 1, 3, 5, \cdots, L-3$

$e) F^\dagger_\infty(v_{L:m}) < r^\dagger_{L:m}\,, m = 1, 2, 3, \cdots, L-1$

$f) r^\dagger_{L:m} < F^\dagger_\infty(v_{L:m+2})\,, m = 1, 2, 3, \cdots, L-3$

Proof

From (5.3.7) and the definition of $r^*_{L:m}$, $q^*_{L:m}$ and $F^*_\infty(x)$, one has $F^*_\infty(v_{L:m}) - r^*_{L:m+1} = \{\alpha_{1:\infty} + \frac{1-\alpha_{1:\infty}}{K}(\frac{1}{a_1} - \frac{1}{a_m})\} - \{\alpha_1 + m\alpha_2\} = \alpha_{1:\infty} - \alpha_1 + \frac{1-\alpha_{1:\infty}}{K}(\frac{1}{a_1} - \frac{1}{a_m}) - m\alpha_2 = \frac{\Delta}{2}\frac{K'+K}{K'(K'-\frac{\Delta}{2})} + \frac{\Delta}{K'}(m - \frac{3}{2}) - \frac{\Delta}{K'-\frac{\Delta}{2}}m = \frac{\Delta}{2}\frac{1}{K'(K'-\frac{\Delta}{2})}\frac{1}{L-\frac{3}{2}}\{K(L-m) - 2K'(L-\frac{3}{2})\} = \frac{\Delta}{2}\frac{1}{K'(K'-\frac{\Delta}{2})}\frac{1}{L-\frac{3}{2}}\{-\frac{1}{a_1}(L + m - 3) - \frac{1}{a_L}(3L - m - 3)\} < 0$ for $m = 1, \cdots, L-1$, where $K' \stackrel{\text{def}}{=} \frac{1}{a_1} + \frac{1}{a_L}$, proving a). One also has $F^*_\infty(v_{L:m}) - r^*_{L:m} = \frac{\Delta}{2}\frac{K'+K}{K'(K'-\frac{\Delta}{2})} + \frac{\Delta}{K'}(m - \frac{3}{2}) - \frac{\Delta}{K'-\frac{\Delta}{2}}(m-1) = \frac{\Delta}{2}\frac{K}{K'(K'-\frac{\Delta}{2})}\frac{L-m}{L-\frac{2}{3}} > 0$, proving b) .

From (5.3.7) and the definition of $r^\sharp_{L:m}$, $q^\sharp_{L:m}$ and $F^*_\infty(x)$, one has, for $m = 1, 3, 5, \cdots, L-1$, $F^*_\infty(v_{L:m}) - r^\sharp_{L:m} = \alpha_{1:\infty} + \frac{1-\alpha_{1:\infty}}{KD}(\frac{1}{c_{high}-c_{mid}} - \frac{1}{v_{L:m}-c_{mid}}) - \frac{4}{4-\alpha_4}(\alpha_3 + \frac{m-1}{2}\alpha_4) = \alpha_{1:\infty} + \frac{1-\alpha_{1:\infty}}{K}(\frac{1}{a_1} - \frac{1}{a_m}) - \frac{4\alpha_3}{4-\alpha_4} - \frac{2(m-1)}{4-\alpha_4}\alpha_4 = \alpha_{1:\infty} - \frac{4\alpha_3}{4-\alpha_4} + \frac{1-\alpha_{1:\infty}}{K}(\frac{1}{a_1} - \frac{1}{a_m}) - \frac{2(m-1)}{4-\alpha_4}\alpha_4 = \frac{K'-K}{K'}\frac{-\Delta}{2K'-\Delta} + \frac{\Delta}{K'}(m - \frac{3}{2}) - \frac{2\Delta(m-1)}{2K'-\Delta} = \frac{\Delta}{K'(2K'-\Delta)}\{-K' + K + (2K'-\Delta)(m-\frac{3}{2}) - 2K'(m-1)\} = \frac{\Delta}{K'(2K'-\Delta)}\{K\frac{L-m}{L-\frac{3}{2}} - 2K'\} < \frac{\Delta}{K'(2K'-\Delta)}\{2K - 2K'\} < 0$, where $K' \stackrel{\text{def}}{=} \frac{1}{a_1} + \frac{1}{a_L}$, proving c). For part d), one also has, for $m = 1, 3, 5, \cdots, L-3$, $F^*_\infty(v_{L:m+2}) - r^\sharp_{L:m} = \alpha_{1:\infty} + \frac{1-\alpha_{1:\infty}}{K}(\frac{1}{a_1} - \frac{1}{a_{m+2}}) - \frac{4\alpha_3}{4-\alpha_4} - \frac{2(m-1)}{4-\alpha_4}\alpha_4 = \frac{K'-K}{K'}\frac{-\Delta}{2K'-\Delta} + \frac{\Delta}{K'}(m + \frac{1}{2}) - \frac{2\Delta(m-1)}{2K'-\Delta} = \frac{\Delta}{K'(2K'-\Delta)}\{K\frac{L-m-2}{L-\frac{3}{2}} + 2K'\} \geq \frac{\Delta}{K'(2K'-\Delta)}\{K\frac{L-(L-3)-2}{L-\frac{3}{2}} + 2K'\} > \frac{\Delta}{K'(2K'-\Delta)}\,2K' > 0$. Next we prove part e). From (5.3.7) and the definition of $r^\dagger_{L:m}$,

$q^{\dagger}_{L:m}$ and $F^{\dagger}_{\infty}(x)$, one has, for $m = 1, 2, 3, \cdots, L-1$, $F^{\dagger}_{\infty}(v_{L:m}) - r^{\dagger}_{L:m} = \frac{1-\alpha_{6:\infty}}{KD}(\frac{1}{c_{high}-c_{mid}} - \frac{1}{v_{L:m}-c_{mid}}) - (m-1)\alpha_5 = \frac{1-(a_1/a_L)}{K}(\frac{1}{a_1} - \frac{1}{a_m}) - (m-1)\Delta a_1 = a_1(\frac{1}{a_1} - \frac{1}{a_m}) - (m-1)\Delta a_1 = a_1\{K - (L-m)\Delta - (m-1)\Delta\} = a_1\{(L - \frac{3}{2}) - (L-1)\Delta\} = -\frac{1}{2}a_1\Delta < 0$. Finally we prove part f). Similarly as in part e), one has, for $m = 1, 2, 3, \cdots, L-3$, $F^{\dagger}_{\infty}(v_{L:m+2}) - r^{\dagger}_{L:m} = \frac{1-\alpha_{6:\infty}}{KD}(\frac{1}{c_{high}-c_{mid}} - \frac{1}{v_{L:m+2}-c_{mid}}) - (m-1)\alpha_5 = \frac{1-(a_1/a_L)}{K}(\frac{1}{a_1} - \frac{1}{a_{m+2}}) - (m-1)\Delta a_1 = a_1(\frac{1}{a_1} - \frac{1}{a_{m+2}}) - (m-1)\Delta a_1 = a_1\{K-(L-m-2)\Delta-(m-1)\Delta\} = a_1\{(L-\frac{3}{2})\Delta-(L-3)\Delta\} = \frac{3}{2}a_1\Delta > 0$.

<u>Proof of Theorem 5.4.2</u>

From the definition of $X^*(\omega)$, one has $P[X^*(\omega) \le x] = P[\omega \in (0, \alpha_{1:\infty}] \vee (\alpha_{1:\infty} < \omega \le F^*_{\infty}(x))] = P[0 < \omega \le F^*_{\infty}(x)]$. Since P is the one-dimensional uniform probability measure on $(\Omega, \mathcal{F})$, one has $P[0 < \omega \le F^*_{\infty}(x)] = F^*_{\infty}(x)$. Similarly one has $P[X^{\dagger}(\omega) \le x] = F^{\dagger}_{\infty}(x)$, proving 1). For part 2), from the definition of $X^*_L(\omega)$, one has $P[X^*_L(\omega) = v_{L:m}] = P[\omega \in \Omega^*_{L:m}] = P[r^*_{L:m-1} < \omega \le r^*_{L:m}] = q^*_{L:m}$. Similarly one has $P[X^{\sharp}_L(\omega) = v_{L:m}] = q^{\sharp}_{L:m}$ and $P[X^{\dagger}_L(\omega) = v_{L:m}] = q^{\dagger}_{L:m}$. In order to prove part 3), we consider the following four cases:

<u>Case1:</u> $\omega \in \Omega^*_{L:1}$

From the definition of $X^*_L(\omega)$ one has $X^*_L(\omega) = v_{L:1}$. From lemma B.8 b), one has $r^*_{L:1} < F^*_{\infty}(v_{L:1}) = \alpha_{1:\infty}$ so that $\Omega^*_{L:1} \subset (0, \alpha_{1:\infty}]$. Then from the definition of $X^*(\omega)$, it is clear that $X^*(\omega) = v_{L:1} = X^*_L(\omega)$.

<u>Case2:</u> $\omega \in \Omega^*_{L:2}$

From the definition of $X^*_L(\omega)$ one has $X^*_L(\omega) = v_{L:2}$. Since $\omega \in \Omega^*_{L:2}$, one has $\omega \le r^*_{L:2}$. It then follows from the definition of $X^*(\omega)$ that $v_{L:1} \le X^*(\omega) \le F^{*-1}_{\infty}(\omega) \le F^{*-1}_{\infty}(r^*_{L:2})$, where the last inequality is yielded since $F^*_{\infty}(x)$ is monotonically increasing. From Lemma B.8 a), one has $F^{*-1}_{\infty}(r^*_{L:2}) < v_{L:3}$. These observations imply that $|X^*(\omega) - X^*_L(\omega)| < |v_{L:3} - v_{L:1}|$.

<u>Case3:</u> $\omega \in \Omega^*_{L:m}$, $m = 3, \cdots, L-1$

From the definition of $X^*_L(\omega)$ one has $X^*_L(\omega) = v_{L:m}$. We note from Lemma B.8 a) that

$\alpha_{1:\infty} = F^*_\infty(v_{L:1}) < r^*_{L:2}$ and it is clear from the condition of this case that $r^*_{L:2} < \omega$ so that $\alpha_{1:\infty} < \omega$. It then follows from this and $\omega \in \Omega^*_{L:m}$ together with the definition of $X^*(\omega)$ that $F^{*-1}_\infty(r^*_{L:m-1}) < X^*(\omega) \leq F^{*-1}_\infty(r^*_{L:m})$. Similarly as in Case2 we have $v_{L:m-2} < F^{*-1}_\infty(r^*_{L:m-1})$ and $F^{*-1}_\infty(r^*_{L:m}) < v_{L:m}$ for $m = 3, \cdots, L-1$. Then we obtain that $|X^*(\omega) - X^*_L(\omega)| < |v_{L:m} - v_{L:m-2}|$ for $m = 3, \cdots, L-1$.

<u>Case4:</u> $\omega \in \Omega^*_{L:L}$

From the definition of $X^*_L(\omega)$, one has $X^*_L(\omega) = v_{L:L}$. Since $\omega \in \Omega^*_{L:L}$, it then follows from the definition of $X^*(\omega)$ that $F^{*-1}_\infty(r^*_{L:L-1}) < X^*(\omega) \leq F^{*-1}_\infty(r^*_{L:L}) = v_{L:L} = U$. From Lemma B.8 a) , we have $v_{L:L-2} < F^{*-1}_\infty(r^*_{L:L-1})$. Then we obtain that $|X^*(\omega) - X^*_L(\omega)| < |v_{L:L} - v_{L:L-2}|$.

Since $|v_{L:m+1} - v_{L:m-1}| < |v_{L:L} - v_{L:L-2}|$ for all $m = 2, \cdots, L-1$, one has that $|X^*(\omega) - X^*_L(\omega)| < |v_{L:L} - v_{L:L-2}|$ for all $\omega \in \Omega$. Since $|v_{L:L} - v_{L:L-2}| \to 0$ as $\tilde{L} \to \infty$, it then follows for all $\omega \in \Omega$ that $X^*_L(\omega) \to X^*(\omega)$ as $\tilde{L} \to \infty$.

The other parts of this Lemma can also be shown in the similar way by using Lemma B.8 c) d) e) f).

<u>**Proof of Lemma 5.4.3**</u>:

We first prove $\lim_{L\to\infty} V_1(Y_{1,L}, X^\dagger_{2,L}) = V_1(Y_1, X^\dagger_2)$ of c) . The other cases can be shown similarly. Let $Z^k,\ k = 1, \cdots, 4$ be defined by $Z^1 = \delta_{\{c_{high} < Y_1 < X^\dagger_2\}}$, $Z^2 = \delta_{\{c_{high} < Y_1 = X^\dagger_2\}}$, $Z^3 = \delta_{\{Y_1 = c_{high}\}}$ and $Z^4 = \delta_{\{X^\dagger_2 = c_{high} < Y_1\}}$. $Z^k_L,\ k = 1, \cdots, 4$ can be defined similarly by replacing Y_1 by $Y_{1,L}$ and $X^\dagger_2$ by $X^\dagger_{2,L}$ respectively. We next prove that $Z^k_L \overset{a.e.}{\to} Z^k, k = 1, \cdots, 4$. The following six cases, as depicted in Figure B.1.1, are considered.

<u>Case1:</u> $(\omega_1, \omega_2) \in \{(\omega_1, \omega_2) | [c_{high} < Y_1(\omega_1)] \wedge [c_{high} < X^\dagger_2(\omega_2)] \wedge [Y_1(\omega_1) > X^\dagger_2(\omega_2)]\}$

Let $\epsilon \overset{\text{def}}{=} Y_1 - X^\dagger_2 > 0$. Since $Y_{1,L} \overset{a.e.}{\to} Y_1$ and $X^\dagger_{2,L} \overset{a.e.}{\to} X^\dagger_2$, one has, for sufficiently large N, that $|Y_1 - Y_{1,L}| < \frac{\epsilon}{3}$ and $|X^\dagger_2 - X^\dagger_{2,L}| < \frac{\epsilon}{3}$ for $L > N$. It then follows that $\frac{\epsilon}{3} = Y_1 - \frac{\epsilon}{3} - X^\dagger_2 - \frac{\epsilon}{3} < Y_1 - (Y_1 - Y_{1,L}) - X^\dagger_2 - (X^\dagger_{2,L} - X^\dagger_2) = Y_{1,L} - X^\dagger_2$ for $L > N$. Finally, these observations

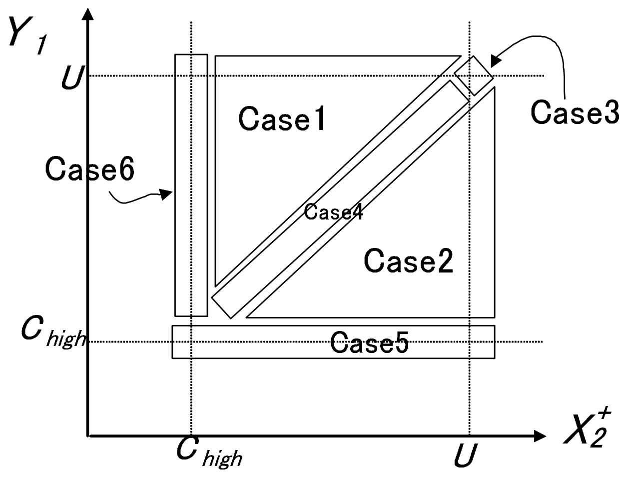

Figure B.1.1: Image of the Cases

imply that

$$Z_L^1 = Z^1 = 0 \quad \text{for } L > N \quad ; \textit{ and} \tag{B.1.6}$$

$$Z_L^2 = Z^2 = 0 \quad \text{for } L > N \quad . \tag{B.1.7}$$

From the condition of the Case1, one sees $c_{high}(= v_{L:1}) < Y_1$. It then follows from the definition of $Y_{1,L}$ that $c_{high} < Y_{1,L}$ for all L. Thus one has $Z^3 = Z_L^3 = 0$ for all L. From the condition of the Case1, one also sees $c_{high} < X_2^\dagger$. Since $\omega_2 \in (0, 1]$, one has $\omega_2 \neq 0$. Hence, from the definition of $X_{2,L}^\dagger$, for sufficiently large N, $c_{high} = v_{L:1} < X_{2,L}^\dagger$, and therefore $Z^4 = Z_L^4 = 0$ for $L > N$.

<u>Case2:</u> $(\omega_1, \omega_2) \in \{(\omega_1, \omega_2)|[c_{high} < Y_1(\omega_1)] \wedge [c_{high} < X_2^\dagger(\omega_2)] \wedge [Y_1(\omega_1) < X_2^\dagger(\omega_2)]\}$

Let N be sufficiently large number. Then in a similar way as in Case1 it is clear that $Z_L^1 = Z^1 = 1$, $Z_L^2 = Z^2 = 0$ for $L > N$; and $Z^3 = Z_L^3 = 0$,$Z^4 = Z_L^4 = 0$ for all L

<u>Case3:</u> $(\omega_1, \omega_2) \in \{(\omega_1, \omega_2)|Y_1(\omega_1) = X_2^\dagger(\omega_2) = U\}$

From the definition of $Y_{1,L}$ in Lemma 5.4.3, one sees $Y_{1,L} = U$ for all L. Since $\Delta > 0$ and $\lim_{\tilde{L}\to\infty} \Delta = 0$, one has $\alpha_{6:\infty} < q_{L:L}^\dagger = \alpha_6 = a_1(\frac{1}{a_L} + \frac{\Delta}{2})$, and consequenly $(1 - \alpha_{6:\infty}, 1] \subset$

$(1-q^{\dagger}_{L:L}, 1] = (r^{\dagger}_{L:L-1}, r^{\dagger}_{L:L}]$. Since $X^{\dagger}_2(\omega_2) = U(= v_{L:L})$, from the definition of $X^{\dagger}_2$, one has $\omega_2 \in (1-\alpha_{6:\infty}, 1]$ so that $\omega_2 \in (r^{\dagger}_{L:L-1}, r^{\dagger}_{L:L}]$. It then follows from the definition of $X^{\dagger}_{2,L}$ that $X^{\dagger}_{2,L} = v_{L:L} = U$ for all L. These observations imply that $[Z^k = Z^k_L = 0, k = 1,3,4$; and $Z^2 = Z^2_L = 1]$ for all$L$.

Case4: $(\omega_1, \omega_2) \in \{(\omega_1, \omega_2) | c_{high} < Y_1(\omega_1) = X^{\dagger}_2(\omega_2) < U\}$

Since it is clear that $P[c_{high} < Y_1(\omega_1) = X^{\dagger}_2(\omega_2) < U] = 0$ we do not have to examin limiting bahavior of $Z^k_L, k = 1, \cdots, 4$.

Case5: $(\omega_1, \omega_2) \in \{(\omega_1, \omega_2) | Y_1(\omega_1) = c_{high}\}$

From the definition of $Y_{1,L}$ one has $Y_{1,L} = c_{high}$ for all L. Hence it is clear that $[Z^k = Z^k_L = 0, k = 1,2,4$; and $Z^3 = Z^3_L = 1]$ for all L

Case6: $(\omega_1, \omega_2) \in \{(\omega_1, \omega_2) | X^{\dagger}_2 = c_{high} < Y_1(\omega_1)\}$

From the definition of $X^{\dagger}_2$, it is clear $P[c_{high} = X^{\dagger}_2(\omega_2)] = 0$, hence we do not have to examine limiting behavior of $Z^k_L, k = 1, \cdots, 4$. From the results of these Cases one obtain that $Z^k_L \overset{a.e.}{\to} Z^k$ as $L \to \infty$ for $k = 1, 2, \cdots, 4$. From (5.2.1), one has $h_1(Y_1, X^{\dagger}_2) = D\{2(Y_1 - c_{mid})Z^1 + (Y_1 - c_{mid})Z^2 + (c_{high} - c_{low})Z^3 + (Y_1 - c_{low})Z^4\}$. It should be noted that $h_1(Y_1, X^{\dagger}_2)$ can be written as the continuous function of $Y_1, X^{\dagger}_2$ and $Z^k, i = 1, \cdots, 4$. According to the a) of this Lemma, $Y_{1,L} \overset{a.e.}{\to} Y_1$, and $Z^k_L \overset{a.e.}{\to} Z^k, k = 1, \cdots, 4$, as we prove above. Hence one concludes that $\lim_{L\to\infty} V_1(Y_{1,L}, X^*_{2,L}) = \lim_{L\to\infty} E[h_1(Y_{1,L}, X^*_{2,L})] = E[h_1(Y_1, X^*_2)] = V_1(Y_1, X^*_2)$. The other parts of this Lemma can also be shown in the similar way, completing the proof.

Appendix C

Publications

- Kazuki Takahashi, Tadashi Ishizaka and Isao Honma, “An Application of Information Theory to the Sensitivity Analysis of Cogeneration System Performance”, *Energy and Resources, 1996 Vol.17, No.1 80–87.*

- Kazuki Takahashi and Tadashi Ishizaka, “Application of Information Theory for the Analysis of Cogeneration-System Performance”, *Applied Energy, Nov.1998 Vol.61, No.3 147–162.*

- Kazuki Takahashi and Ushio Sumita, “On Non-Existence of Nash Equilibrium of M Person Game with Pure Strategy for Delivery Services”, *Proceedings of the 2009 IEEE International conference on Industrial Engineering and Engineering Management(2009) .*

- Kazuki Takahashi and Ushio Sumita, “Structural Analysis of Two Person Game with Mixed Strategy for Delivery Services - How to Decide Service Area and Pricing Strategy - ”, *Journal of Japan Industrial Management Association, Feb.2011 Vol.61, No.6 314–324* (in Japanese).